MES w statyce ośrodka ciągłego

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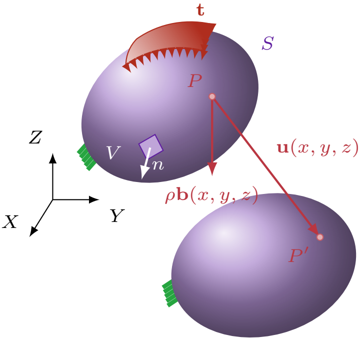
Zakres prezentacji

Stan równowagi



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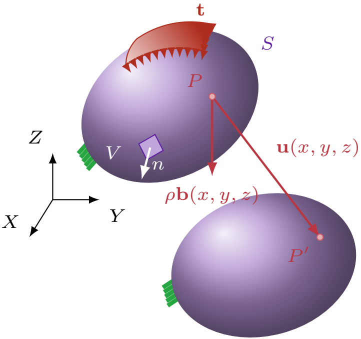

Wektor gęstości sił masowych [N/m^3]

$$\rho\mathbf{b} = \rho \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$$

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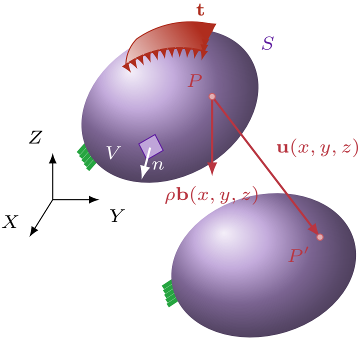

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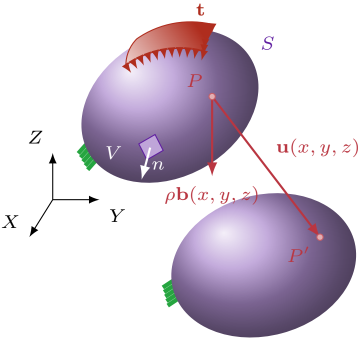

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Przemieszczenie, odkształcenie, naprężenie (notacja Voigta)

$$\mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}, \quad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix} \rightarrow \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}, \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \rightarrow \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}$$

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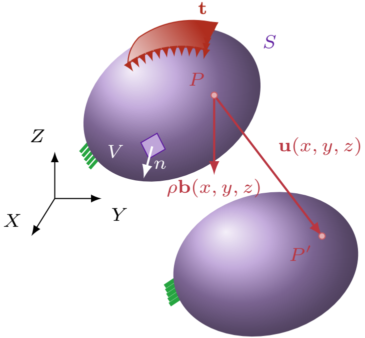

Równanie równowagi ciała

$$\int_S \mathbf{t}\mathrm{d}S + \int_V \rho\mathbf{b}\mathrm{d}V = 0$$

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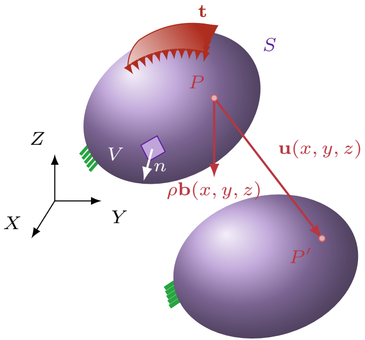

Równanie równowagi ciała

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Statyczne warunki brzegowe

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$$

gdzie $\boldsymbol{\sigma}$ – tensor naprężenia

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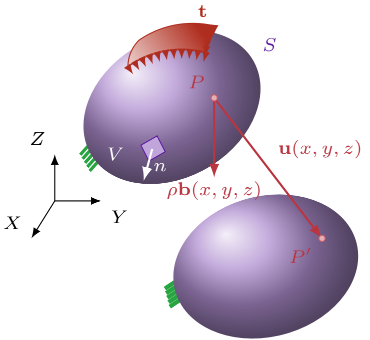

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Wykorzystując twierdzenie Greena–Gaussa–Ostrogradzkiego

$$\int_S \boldsymbol{\sigma}\mathbf{n}\,\mathrm{d}S = \int_V \mathbf{L}^{\mathrm{T}}\boldsymbol{\sigma}\,\mathrm{d}V$$

gdzie $\mathbf{L}$ – macierz operatorów różniczkowych

Równania równowagi

Równania Naviera

$$\int_V \left(\mathbf{L}^{\mathrm{T}}\boldsymbol{\sigma} + \rho\mathbf{b}\right) \mathrm{d}V = 0 \Longleftrightarrow \mathbf{L}^{\mathrm{T}}\boldsymbol{\sigma} + \rho\mathbf{b} = 0 \quad \forall P \in V$$

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Sformułowanie słabe – funkcja wagowa $w \equiv \delta\mathbf{u}$ – kinematycznie dopuszczalna wariacja przemieszczenia

$$\int_V (\delta\mathbf{u})^{\mathrm{T}} \left(\mathbf{L}^{\mathrm{T}}\boldsymbol{\sigma} + \rho\mathbf{b}\right) \mathrm{d}V = 0 \quad \forall\delta\mathbf{u}$$

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$$\underbrace{\boxed{\int_V (\mathbf{L}\delta\mathbf{u})^{\mathrm{T}}\boldsymbol{\sigma}\mathrm{d}V}}_{\text{praca sił wewnętrznych}} = \underbrace{\boxed{\int_S (\delta\mathbf{u})^{\mathrm{T}}\mathbf{t}\mathrm{d}S + \int_V (\delta\mathbf{u})^{\mathrm{T}}\rho\mathbf{b}\mathrm{d}V}}_{\text{praca sił zewnętrznych}}$$

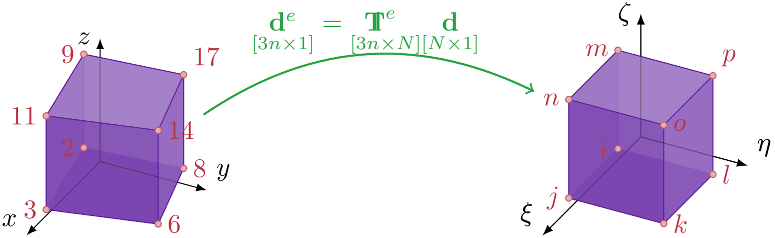
Dyskretyzacja MES (n=LWE, N=LSSU, E=LEU)

Aproksymacja pola przemieszczeń

$$\mathbf{u}^{eh} = \sum_{i=1}^{n} N_i^e(\xi,\eta,\zeta)\mathbf{d}_i^e = \mathbf{N}^e\mathbf{d}^e$$

$$\underset{[3\times 3n]}{\mathbf{N}^e} = \left[\begin{array}{ccc|c|ccc} N_1^e & 0 & 0 & \dots & N_n^e & 0 & 0 \\ 0 & N_1^e & 0 & \dots & 0 & N_n^e & 0 \\ 0 & 0 & N_1^e & \dots & 0 & 0 & N_n^e \end{array}\right] \qquad \underset{[3n\times 1]}{\mathbf{d}^e} = \begin{bmatrix} \mathbf{d}_1^e \\ \dots \\ \mathbf{d}_n^e \end{bmatrix}$$

$$\underset{[3n\times 1]}{\mathbf{d}^e} = \underset{[3n\times N]}{\mathbb{T}^e}\ \underset{[N\times 1]}{\mathbf{d}}$$



$\mathbb{T}^e = \mathbf{T}^e\,\mathbb{B}^e$ – macierz transformacji uwzględniająca topologię ($\mathbb{B}^e$) oraz cosinusy kierunkowe pomiędzy osiami układu globalnego i lokalnego ($\mathbf{T}^e$)

Równanie równowagi dla układu zdyskretyzowanego

Równanie równowagi ($\rho \mathbf{b}^e = \mathbf{f}^e$ – wektor sił objętościowych)

$$\sum_{e=1}^{E} \left\{ \int_{V^e} (\mathbf{L}^e \delta \mathbf{u}^e)^{\mathrm{T}} \boldsymbol{\sigma}^e \mathrm{d}V^e - \int_{S^e} (\delta \mathbf{u}^e)^{\mathrm{T}} \mathbf{t}^e \mathrm{d}S^e - \int_{V^e} (\delta \mathbf{u}^e)^{\mathrm{T}} \mathbf{f}^e \mathrm{d}V^e \right\} = 0$$

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$$\sum_{e=1}^{E}\left\{\int_{V^e}(\mathbf{L}^e\delta\mathbf{u}^e)^{\mathrm{T}}\boldsymbol{\sigma}^e\mathrm{d}V^e-\int_{S^e}(\delta\mathbf{u}^e)^{\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e-\int_{V^e}(\delta\mathbf{u}^e)^{\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e\right\}=0$$

$$\sum_{e=1}^{E}\left\{\int_{V^e}(\mathbf{B}^e\delta\mathbf{d}^e)^{\mathrm{T}}\boldsymbol{\sigma}^e\mathrm{d}V^e-\int_{S^e}(\mathbf{N}^e\delta\mathbf{d}^e)^{\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e-\int_{V^e}(\mathbf{N}^e\delta\mathbf{d}^e)^{\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e\right\}=0$$

$$\sum_{e=1}^{E}(\mathbb{T}^e\delta\mathbf{d})^{\mathrm{T}}\left\{\int_{V^e}\mathbf{B}^{e\mathrm{T}}\boldsymbol{\sigma}^e\mathrm{d}V^e-\int_{S^e}\mathbf{N}^{e\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e-\int_{V^e}\mathbf{N}^{e\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e\right\}=0$$

$$\sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\left\{\int_{V^e}\mathbf{B}^{e\mathrm{T}}\boldsymbol{\sigma}^e\mathrm{d}V^e-\int_{S^e}\mathbf{N}^{e\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e-\int_{V^e}\mathbf{N}^{e\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e\right\}=0$$

Równanie równowagi dla układu zdyskretyzowanego

Równanie równowagi

$$\sum_{e=1}^{E}\left\{\int_{V^e}(\mathbf{L}^e\delta\mathbf{u}^e)^{\mathrm{T}}\boldsymbol{\sigma}^e\mathrm{d}V^e-\int_{S^e}(\delta\mathbf{u}^e)^{\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e-\int_{V^e}(\delta\mathbf{u}^e)^{\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e\right\}=0$$

$$\sum_{e=1}^{E}\left\{\int_{V^e}(\mathbf{B}^e\delta\mathbf{d}^e)^{\mathrm{T}}\boldsymbol{\sigma}^e\mathrm{d}V^e-\int_{S^e}(\mathbf{N}^e\delta\mathbf{d}^e)^{\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e-\int_{V^e}(\mathbf{N}^e\delta\mathbf{d}^e)^{\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e\right\}=0$$

$$\sum_{e=1}^{E}(\mathbb{T}^e\delta\mathbf{d})^{\mathrm{T}}\left\{\int_{V^e}\mathbf{B}^{e\mathrm{T}}\boldsymbol{\sigma}^e\mathrm{d}V^e-\int_{S^e}\mathbf{N}^{e\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e-\int_{V^e}\mathbf{N}^{e\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e\right\}=0$$

$$\sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\left\{\int_{V^e}\mathbf{B}^{e\mathrm{T}}\boldsymbol{\sigma}^e\mathrm{d}V^e-\int_{S^e}\mathbf{N}^{e\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e-\int_{V^e}\mathbf{N}^{e\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e\right\}=0$$

$$\sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\left\{\int_{V^e}\mathbf{B}^{e\mathrm{T}}\boldsymbol{\sigma}^e\mathrm{d}V^e\right\}=\sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\left\{\int_{S^e}\mathbf{N}^{e\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e+\int_{V^e}\mathbf{N}^{e\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e\right\}$$

Równanie równowagi dla układu zdyskretyzowanego

Równanie równowagi

$$\sum_{e=1}^{E}\left\{\int_{V^e}(\mathbf{L}^e\delta\mathbf{u}^e)^{\mathrm{T}}\boldsymbol{\sigma}^e\mathrm{d}V^e-\int_{S^e}(\delta\mathbf{u}^e)^{\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e-\int_{V^e}(\delta\mathbf{u}^e)^{\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e\right\}=0$$

$$\sum_{e=1}^{E}\left\{\int_{V^e}(\mathbf{B}^e\delta\mathbf{d}^e)^{\mathrm{T}}\boldsymbol{\sigma}^e\mathrm{d}V^e-\int_{S^e}(\mathbf{N}^e\delta\mathbf{d}^e)^{\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e-\int_{V^e}(\mathbf{N}^e\delta\mathbf{d}^e)^{\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e\right\}=0$$

$$\sum_{e=1}^{E}(\mathbb{T}^e\delta\mathbf{d})^{\mathrm{T}}\left\{\int_{V^e}\mathbf{B}^{e\mathrm{T}}\boldsymbol{\sigma}^e\mathrm{d}V^e-\int_{S^e}\mathbf{N}^{e\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e-\int_{V^e}\mathbf{N}^{e\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e\right\}=0$$

$$\sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\left\{\int_{V^e}\mathbf{B}^{e\mathrm{T}}\boldsymbol{\sigma}^e\mathrm{d}V^e-\int_{S^e}\mathbf{N}^{e\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e-\int_{V^e}\mathbf{N}^{e\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e\right\}=0$$

$$\sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\left\{\int_{V^e}\mathbf{B}^{e\mathrm{T}}\boldsymbol{\sigma}^e\mathrm{d}V^e\right\}=\sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\left\{\int_{S^e}\mathbf{N}^{e\mathrm{T}}\mathbf{t}^e\mathrm{d}S^e+\int_{V^e}\mathbf{N}^{e\mathrm{T}}\mathbf{f}^e\mathrm{d}V^e\right\}$$

siły wewnętrzne = siły zewnętrzne

Równanie równowagi dla układu zdyskretyzowanego

Równanie równowagi

$$\sum_{e=1}^{E} \mathbf{T}^{e\,\mathrm{T}} \left\{ \int_{V^e} \mathbf{B}^{e\,\mathrm{T}} \boldsymbol{\sigma}^e \mathrm{d}V^e \right\} = \sum_{e=1}^{E} \mathbf{T}^{e\,\mathrm{T}} \left\{ \int_{S^e} \mathbf{N}^{e\,\mathrm{T}} \mathbf{t}^e \mathrm{d}S^e + \int_{V^e} \mathbf{N}^{e\,\mathrm{T}} \mathbf{f}^e \mathrm{d}V^e \right\}$$

Równanie równowagi dla układu zdyskretyzowanego

Równanie równowagi

$$\sum_{e=1}^{E} \mathbf{T}^{e\mathrm{T}} \left\{ \int_{V^e} \mathbf{B}^{e\mathrm{T}} \boldsymbol{\sigma}^e \mathrm{d}V^e \right\} = \sum_{e=1}^{E} \mathbf{T}^{e\mathrm{T}} \left\{ \int_{S^e} \mathbf{N}^{e\mathrm{T}} \mathbf{t}^e \mathrm{d}S^e + \int_{V^e} \mathbf{N}^{e\mathrm{T}} \mathbf{f}^e \mathrm{d}V^e \right\}$$

Uwzględnienie związków kinematycznych i konstytutywnych

liniowa sprężystość: $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$
liniowy związek kinematyczny: $\boldsymbol{\varepsilon} = \mathbf{L}\mathbf{u}$

$$\boldsymbol{\sigma}^e = \mathbf{D}^e \mathbf{L}^e \mathbf{u}^e = \mathbf{D}^e \mathbf{L}^e \mathbf{N}^e \mathbf{d}^e = \mathbf{D}^e \mathbf{B}^e \mathbf{T}^e \mathbf{d}$$

Równanie równowagi dla układu zdyskretyzowanego

Równanie równowagi

$$\sum_{e=1}^{E} \mathbb{T}^{e\,\mathrm{T}} \left\{ \int_{V^e} \mathbf{B}^{e\,\mathrm{T}} \boldsymbol{\sigma}^e \mathrm{d}V^e \right\} = \sum_{e=1}^{E} \mathbb{T}^{e\,\mathrm{T}} \left\{ \int_{S^e} \mathbf{N}^{e\,\mathrm{T}} \mathbf{t}^e \mathrm{d}S^e + \int_{V^e} \mathbf{N}^{e\,\mathrm{T}} \mathbf{f}^e \mathrm{d}V^e \right\}$$

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$$\boldsymbol{\sigma}^e = \mathbf{D}^e \mathbf{L}^e \mathbf{u}^e = \mathbf{D}^e \mathbf{L}^e \mathbf{N}^e \mathbf{d}^e = \mathbf{D}^e \mathbf{B}^e \mathbb{T}^e \mathbf{d}$$

Równanie równowagi

$$\sum_{e=1}^{E} \mathbb{T}^{e\,\mathrm{T}} \left\{ \int_{V^e} \mathbf{B}^{e\,\mathrm{T}} \mathbf{D}^e \mathbf{B}^e \mathbb{T}^e \mathbf{d} \, \mathrm{d}V^e \right\} = \sum_{e=1}^{E} \mathbb{T}^{e\,\mathrm{T}} \left\{ \int_{S^e} \mathbf{N}^{e\,\mathrm{T}} \mathbf{t}^e \mathrm{d}S^e + \int_{V^e} \mathbf{N}^{e\,\mathrm{T}} \mathbf{f}^e \mathrm{d}V^e \right\}$$

Równanie równowagi dla układu zdyskretyzowanego

Równanie równowagi

$$\sum_{e=1}^{E} \mathbb{T}^{e\,\mathrm{T}} \left\{ \int_{V^e} \mathbf{B}^{e\,\mathrm{T}} \boldsymbol{\sigma}^e \mathrm{d}V^e \right\} = \sum_{e=1}^{E} \mathbb{T}^{e\,\mathrm{T}} \left\{ \int_{S^e} \mathbf{N}^{e\,\mathrm{T}} \mathbf{t}^e \mathrm{d}S^e + \int_{V^e} \mathbf{N}^{e\,\mathrm{T}} \mathbf{f}^e \mathrm{d}V^e \right\}$$

Uwzględnienie związków kinematycznych i konstytutywnych

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liniowy związek kinematyczny: $\boldsymbol{\varepsilon} = \mathbf{L}\mathbf{u}$

$$\boldsymbol{\sigma}^e = \mathbf{D}^e \mathbf{L}^e \mathbf{u}^e = \mathbf{D}^e \mathbf{L}^e \mathbf{N}^e \mathbf{d}^e = \mathbf{D}^e \mathbf{B}^e \mathbb{T}^e \mathbf{d}$$

Równanie równowagi

$$\sum_{e=1}^{E} \mathbb{T}^{e\,\mathrm{T}} \left\{ \underbrace{\int_{V^e} \mathbf{B}^{e\,\mathrm{T}} \mathbf{D}^e \mathbf{B}^e \mathrm{d}V^e}_{\bar{\mathbf{K}}^e} \right\} \mathbb{T}^e \mathbf{d} = \sum_{e=1}^{E} \mathbb{T}^{e\,\mathrm{T}} \left\{ \underbrace{\int_{S^e} \mathbf{N}^{e\,\mathrm{T}} \mathbf{t}^e \mathrm{d}S^e}_{\bar{\mathbf{p}}_{\mathrm{b}}^e} + \underbrace{\int_{V^e} \mathbf{N}^{e\,\mathrm{T}} \mathbf{f}^e \mathrm{d}V^e}_{\bar{\mathbf{p}}^e} \right\}$$

Równanie równowagi dla układu zdyskretyzowanego

Równanie równowagi

$$\sum_{e=1}^{E} \mathbf{T}^{e\mathrm{T}} \left\{ \int_{V^e} \mathbf{B}^{e\mathrm{T}} \boldsymbol{\sigma}^e \mathrm{d}V^e \right\} = \sum_{e=1}^{E} \mathbf{T}^{e\mathrm{T}} \left\{ \int_{S^e} \mathbf{N}^{e\mathrm{T}} \mathbf{t}^e \mathrm{d}S^e + \int_{V^e} \mathbf{N}^{e\mathrm{T}} \mathbf{f}^e \mathrm{d}V^e \right\}$$

Uwzględnienie związków kinematycznych i konstytutywnych

liniowa sprężystość: $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$

liniowy związek kinematyczny: $\boldsymbol{\varepsilon} = \mathbf{L}\mathbf{u}$

$$\boldsymbol{\sigma}^e = \mathbf{D}^e \mathbf{L}^e \mathbf{u}^e = \mathbf{D}^e \mathbf{L}^e \mathbf{N}^e \mathbf{d}^e = \mathbf{D}^e \mathbf{B}^e \mathbf{T}^e \mathbf{d}$$

Równanie równowagi

$$\sum_{e=1}^{E} \mathbf{T}^{e\mathrm{T}} \bar{\mathbf{K}}^e \mathbf{T}^e \mathbf{d} = \sum_{e=1}^{E} \mathbf{T}^{e\mathrm{T}} \bar{\mathbf{p}}_{\mathsf{b}}^e + \sum_{e=1}^{E} \mathbf{T}^{e\mathrm{T}} \bar{\mathbf{p}}^e$$

Równanie równowagi dla układu zdyskretyzowanego

Równanie równowagi

$$\sum_{e=1}^{E} \mathbb{T}^{e\mathrm{T}} \left\{ \int_{V^e} \mathbf{B}^{e\mathrm{T}} \boldsymbol{\sigma}^e \mathrm{d}V^e \right\} = \sum_{e=1}^{E} \mathbb{T}^{e\mathrm{T}} \left\{ \int_{S^e} \mathbf{N}^{e\mathrm{T}} \mathbf{t}^e \mathrm{d}S^e + \int_{V^e} \mathbf{N}^{e\mathrm{T}} \mathbf{f}^e \mathrm{d}V^e \right\}$$

Uwzględnienie związków kinematycznych i konstytutywnych

liniowa sprężystość: $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$

liniowy związek kinematyczny: $\boldsymbol{\varepsilon} = \mathbf{L}\mathbf{u}$

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Równanie równowagi

$$\underbrace{\sum_{e=1}^{E} \mathbb{T}^{e\mathrm{T}} \bar{\mathbf{K}}^e \mathbb{T}^e}_{\mathbf{K}} \mathbf{d} = \underbrace{\sum_{e=1}^{E} \mathbb{T}^{e\mathrm{T}} \bar{\mathbf{p}}_{\mathrm{b}}^e}_{\mathbf{p}_{\mathrm{b}}} + \underbrace{\sum_{e=1}^{E} \mathbb{T}^{e\mathrm{T}} \bar{\mathbf{p}}^e}_{\mathbf{p}}$$

Równanie równowagi dla układu zdyskretyzowanego

Równanie równowagi

$$\sum_{e=1}^{E} \mathbb{T}^{e\mathrm{T}} \left\{ \int_{V^e} \mathbf{B}^{e\mathrm{T}} \boldsymbol{\sigma}^e \mathrm{d}V^e \right\} = \sum_{e=1}^{E} \mathbb{T}^{e\mathrm{T}} \left\{ \int_{S^e} \mathbf{N}^{e\mathrm{T}} \mathbf{t}^e \mathrm{d}S^e + \int_{V^e} \mathbf{N}^{e\mathrm{T}} \mathbf{f}^e \mathrm{d}V^e \right\}$$

Uwzględnienie związków kinematycznych i konstytutywnych

liniowa sprężystość: $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$
liniowy związek kinematyczny: $\boldsymbol{\varepsilon} = \mathbf{L}\mathbf{u}$

$$\boldsymbol{\sigma}^e = \mathbf{D}^e \mathbf{L}^e \mathbf{u}^e = \mathbf{D}^e \mathbf{L}^e \mathbf{N}^e \mathbf{d}^e = \mathbf{D}^e \mathbf{B}^e \mathbb{T}^e \mathbf{d}$$

Równanie równowagi

$$\mathbf{K}\mathbf{d} = \mathbf{p}_{\mathsf{b}} + \mathbf{p}$$

Płaski stan naprężenia ($\sigma_z = 0$)

Wektor funkcji przemieszczeń

$$\mathbf{u} = \{u(x,y), v(x,y)\}$$

Wektor odkształcenia

$$\boldsymbol{\varepsilon} = \{\varepsilon_x, \varepsilon_y, \gamma_{xy}\}$$

Wektor naprężenia

$$\boldsymbol{\sigma} = \{\sigma_x, \sigma_y, \tau_{xy}\}$$

Wektor intensywności sił powierzchniowych

$$\mathbf{t} = \{t_x, t_y\}$$

Wektor obciążenia objętościowego

$$\mathbf{f} = \{f_x, f_y\}$$

Macierz związków konstytutywnych

$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Macierz operatorów różniczkowych

$$\mathbf{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

Płaski stan naprężenia ($\sigma_z = 0$)

Macierz sztywności

$$\mathbf{k}^e = \int_{A^e} \mathbf{B}^{e\mathrm{T}} \mathbf{D}^e \mathbf{B}^e h^e \mathrm{d}A^e$$

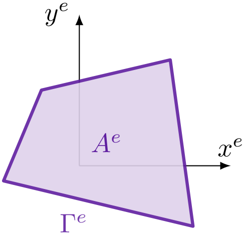
A^e, h^e – pole powierzchni i grubość ES

Wektor obciążenia elementu

$$\mathbf{p}^e = \int_{A^e} \mathbf{N}^{e\mathrm{T}} \mathbf{f}^e h^e \mathrm{d}A^e$$

Wektor sił brzegowych

$$\mathbf{p}_{\mathsf{b}}^e = \int_{\Gamma^e} \mathbf{N}^{e\mathrm{T}} \mathbf{t}^e h^e \mathrm{d}\Gamma^e$$

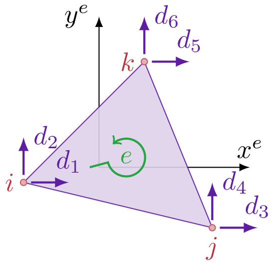


Elementy skończone dla tarczy

Element trójwęzłowy

$$\mathbf{u}^e(x,y) = \mathbf{N}^e(x,y)\,\mathbf{d}^e$$

$$\mathbf{N}^e = \begin{bmatrix} N_i^e & 0 & N_j^e & 0 & N_k^e & 0 \\ 0 & N_i^e & 0 & N_j^e & 0 & N_k^e \end{bmatrix}, \quad \mathbf{d}^e = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix}$$



Elementy skończone dla tarczy

Element trójwęzłowy

$$\mathbf{u}^e(x,y) = \mathbf{N}^e(x,y)\,\mathbf{d}^e$$

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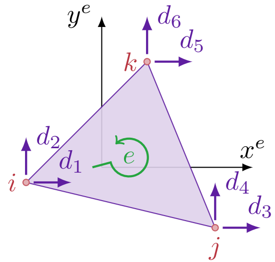
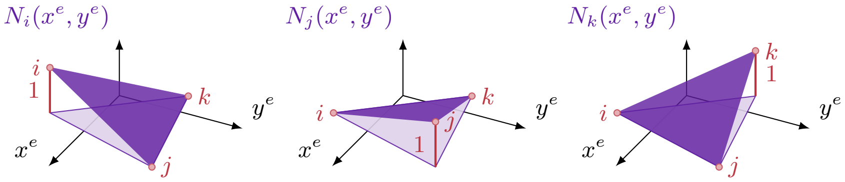




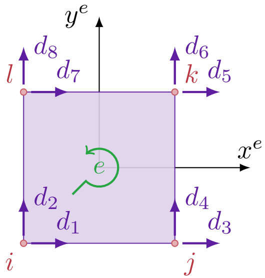
Tarczowe elementy skończone

Element czterowęzłowy

$$\mathbf{u}^e(x,y) = \mathbf{N}^e(x,y)\,\mathbf{d}^e$$

$$\mathbf{N}^e = \begin{bmatrix} N_i^e & 0 & N_j^e & 0 & N_k^e & 0 & N_l^e & 0 \\ 0 & N_i^e & 0 & N_j^e & 0 & N_k^e & 0 & N_l^e \end{bmatrix}$$

$$\mathbf{d}^e = \{d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8\}$$



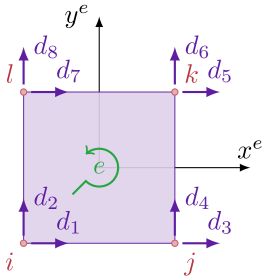
Tarczowe elementy skończone

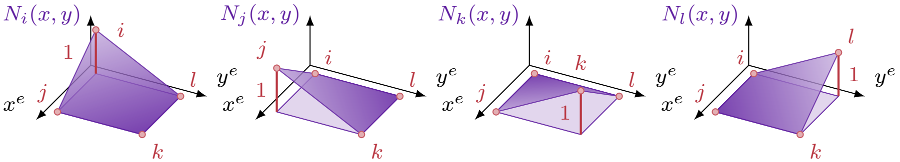
Element czterowęzłowy

$$\mathbf{u}^e(x,y) = \mathbf{N}^e(x,y)\,\mathbf{d}^e$$

$$\mathbf{N}^e = \begin{bmatrix} N_i^e & 0 & N_j^e & 0 & N_k^e & 0 & N_l^e & 0 \\ 0 & N_i^e & 0 & N_j^e & 0 & N_k^e & 0 & N_l^e \end{bmatrix}$$

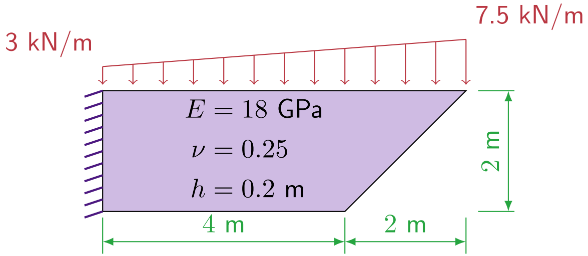
$$\mathbf{d}^e = \{d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8\}$$





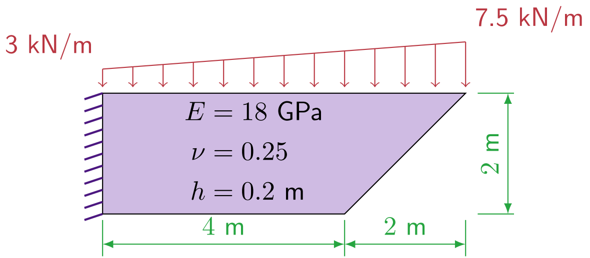
Przykład

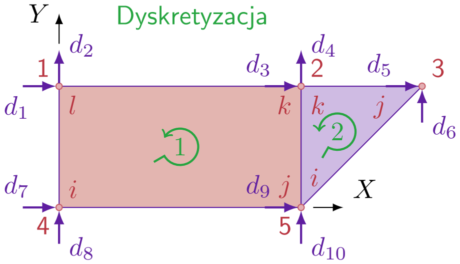
Statyka tarczy



Przykład

Statyka tarczy

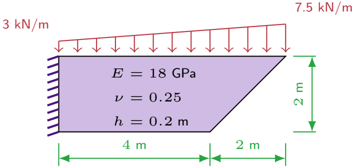


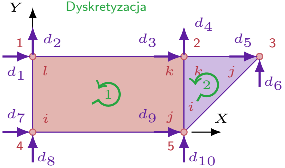


nr elem.	nr węzłów			
1	4	5	2	1
2	5	3	2	

Macierz związków konstytutywnych

$$
\mathbf{D} = \frac{18 \cdot 10^6}{1 - 0.25^2} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix} \text{ [kPa]}
$$





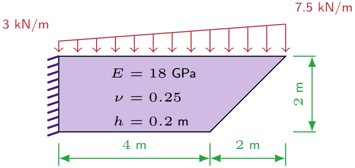
Przykład

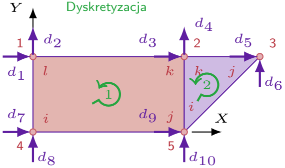
Statyka tarczy

Macierz związków konstytutywnych

$$\mathbf{D} = \frac{18 \cdot 10^6}{1 - 0.25^2} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix} \text{ [kPa]}$$

$$\mathbf{D} = \begin{bmatrix} 19.2 & 4.8 & 0 \\ 4.8 & 19.2 & 0 \\ 0 & 0 & 7.2 \end{bmatrix} \cdot 10^6 \text{ [kPa]}$$



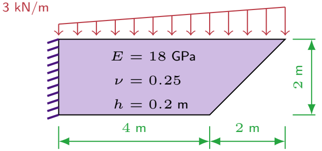


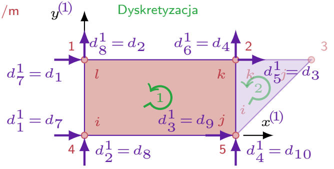
Funkcje kształtu– Element 1

$$N_i^1(x^{(1)},y^{(1)}) = \frac{x^{(1)}y^{(1)} - 2x^{(1)} - 4y^{(1)} + 8}{8}, \qquad N_k^1(x^{(1)},y^{(1)}) = \frac{x^{(1)}y^{(1)}}{8}$$

$$N_j^1(x^{(1)},y^{(1)}) = -\frac{x^{(1)}y^{(1)} - 2x^{(1)}}{8}, \qquad N_l^1(x^{(1)},y^{(1)}) = -\frac{x^{(1)}y^{(1)} - 4y^{(1)}}{8}$$

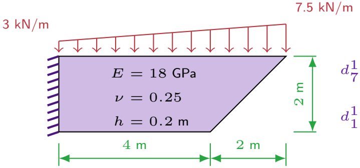
$$\mathbf{N}^1 = \begin{bmatrix} N_i^1 & 0 & N_j^1 & 0 & N_k^1 & 0 & N_l^1 & 0 \\ 0 & N_i^1 & 0 & N_j^1 & 0 & N_k^1 & 0 & N_l^1 \end{bmatrix}$$

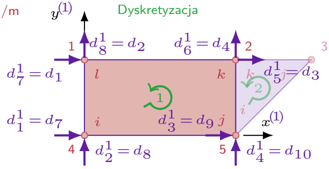




Macierz $\mathbf{K}$ – Element 1

$$
\mathbf{B}^1(x^{(1)}, y^{(1)}) = \begin{bmatrix}
\frac{y^{(1)}}{8} - \frac{1}{4} & 0 & \frac{1}{4} - \frac{y^{(1)}}{8} & 0 & \frac{y^{(1)}}{8} & 0 & -\frac{y^{(1)}}{8} & 0 \\
0 & \frac{x^{(1)}}{8} - \frac{1}{2} & 0 & -\frac{x^{(1)}}{8} & 0 & \frac{x^{(1)}}{8} & 0 & \frac{1}{2} - \frac{x^{(1)}}{8} \\
\frac{x^{(1)}}{8} - \frac{1}{2} & \frac{y^{(1)}}{8} - \frac{1}{4} & -\frac{x^{(1)}}{8} & \frac{1}{4} - \frac{y^{(1)}}{8} & \frac{x^{(1)}}{8} & \frac{y^{(1)}}{8} & \frac{1}{2} - \frac{x^{(1)}}{8} & -\frac{y^{(1)}}{8}
\end{bmatrix}
$$

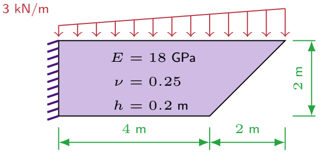


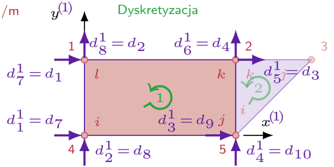


Macierz $\mathbf{K}$ – Element 1

$$\mathbf{B}^1(x^{(1)},y^{(1)}) = \begin{bmatrix} \frac{y^{(1)}}{8}-\frac{1}{4} & 0 & \frac{1}{4}-\frac{y^{(1)}}{8} & 0 & \frac{y^{(1)}}{8} & 0 & -\frac{y^{(1)}}{8} & 0 \\ 0 & \frac{x^{(1)}}{8}-\frac{1}{2} & 0 & -\frac{x^{(1)}}{8} & 0 & \frac{x^{(1)}}{8} & 0 & \frac{1}{2}-\frac{x^{(1)}}{8} \\ \frac{x^{(1)}}{8}-\frac{1}{2} & \frac{y^{(1)}}{8}-\frac{1}{4} & -\frac{x^{(1)}}{8} & \frac{1}{4}-\frac{y^{(1)}}{8} & \frac{x^{(1)}}{8} & \frac{y^{(1)}}{8} & \frac{1}{2}-\frac{x^{(1)}}{8} & -\frac{y^{(1)}}{8} \end{bmatrix}$$

$$\mathbf{K}^1 = \int_0^2\int_0^4 \mathbf{B}^{1\,\mathrm{T}}\mathbf{D}\mathbf{B}^1 h\,\mathrm{d}x^{(1)}\mathrm{d}y^{(1)} = \begin{bmatrix} 16 & 6 & -1.6 & -1.2 & -8 & -6 & -6.4 & 1.2 \\ 6 & 28 & 1.2 & 10.4 & -6 & -14 & -1.2 & -24.4 \\ -1.6 & 1.2 & 16 & -6 & -6.4 & -1.2 & -8 & 6 \\ -1.2 & 10.4 & -6 & 28 & 1.2 & -24.4 & 6 & -14 \\ -8 & -6 & -6.4 & 1.2 & 16 & 6 & -1.6 & -1.2 \\ -6 & -14 & -1.2 & -24.4 & 6 & 28 & 1.2 & 10.4 \\ -6.4 & -1.2 & -8 & 6 & -1.6 & 1.2 & 16 & -6 \\ 1.2 & -24.4 & 6 & -14 & -1.2 & 10.4 & -6 & 28 \end{bmatrix} \cdot 10^5$$



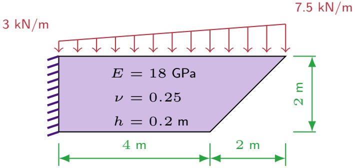


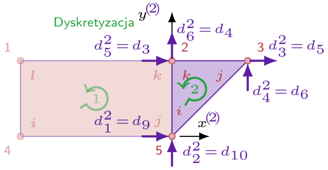
Funkcje kształtu – Element 2

$$N_i^2(x^{(2)}, y^{(2)}) = \frac{2 - y^{(2)}}{2}, \qquad N_k^2(x^{(2)}, y^{(2)}) = \frac{y^{(2)} - x^{(2)}}{2}$$

$$N_j^2(x^{(2)}, y^{(2)}) = \frac{x^{(2)}}{2}$$

$$\mathbf{N}^2 = \begin{bmatrix} N_i^2 & 0 & N_j^2 & 0 & N_k^2 & 0 \\ 0 & N_i^2 & 0 & N_j^2 & 0 & N_k^2 \end{bmatrix}$$



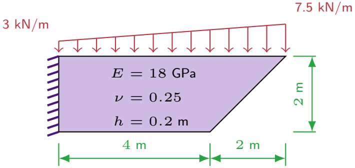


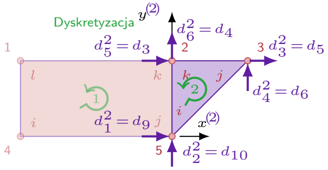
Przykład

Statyka tarczy

Macierz $\mathbf{K}$ – Element 2

$$\mathbf{B}^2(x^{(2)}, y^{(2)}) = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

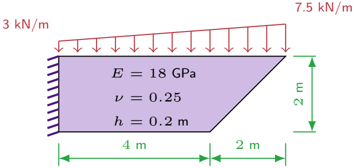


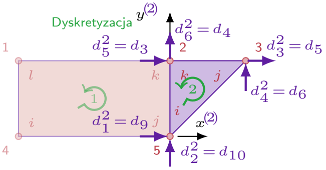


Macierz $\mathbf{K}$ – Element 2

$$\mathbf{B}^2(x^{(2)}, y^{(2)}) = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

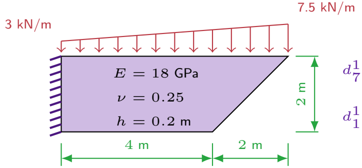
$$\mathbf{K}^2 = \mathbf{B}^{2\,\mathrm{T}} \mathbf{D} \mathbf{B}^2 h A^2 = \begin{bmatrix} 7.2 & 0 & 0 & -7.2 & -7.2 & 7.2 \\ 0 & 19.2 & -4.8 & 0 & 4.8 & -19.2 \\ 0 & -4.8 & 19.2 & 0 & -19.2 & 4.8 \\ -7.2 & 0 & 0 & 7.2 & 7.2 & -7.2 \\ -7.2 & 4.8 & -19.2 & 7.2 & 26.4 & -12 \\ 7.2 & -19.2 & 4.8 & -7.2 & -12 & 26.4 \end{bmatrix} \cdot 10^5$$

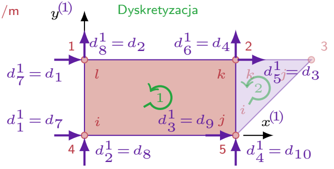




Wektor $\mathbf{p}_\mathsf{b}$ – Element 1

$$\mathbf{p}_\mathsf{b}^1 = \int_{\Gamma^1_{ij}} {\mathbf{N}^1}^\mathrm{T} \mathbf{t} \mathrm{d}\Gamma + \int_{\Gamma^1_{jk}} {\mathbf{N}^1}^\mathrm{T} \mathbf{t} \mathrm{d}\Gamma + \int_{\Gamma^1_{kl}} {\mathbf{N}^1}^\mathrm{T} \mathbf{t} \mathrm{d}\Gamma + \int_{\Gamma^1_{li}} {\mathbf{N}^1}^\mathrm{T} \mathbf{t} \mathrm{d}\Gamma$$

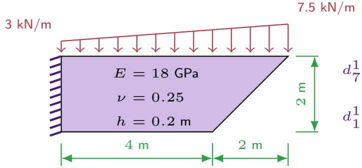


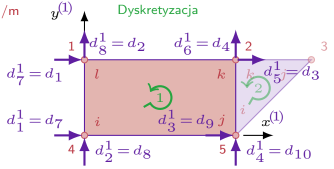


Wektor $\mathbf{p}_\mathrm{b}$ – Element 1

$$\mathbf{p}_\mathrm{b}^1 = \int_{\Gamma^1_{ij}} \mathbf{N}^{1^\mathrm{T}} \underbrace{\mathbf{t}}^{w.b.\,=\,0} \mathrm{d}\Gamma + \int_{\Gamma^1_{jk}} \mathbf{N}^{1^\mathrm{T}} \mathbf{t} \mathrm{d}\Gamma + \int_{\Gamma^1_{kl}} \mathbf{N}^{1^\mathrm{T}} \mathbf{t} \mathrm{d}\Gamma + \int_{\Gamma^1_{li}} \mathbf{N}^{1^\mathrm{T}} \mathbf{t} \mathrm{d}\Gamma$$

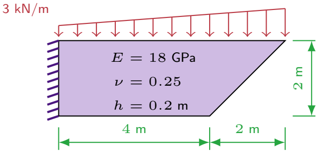
wspólna krawędź
równowaga sił
wzdłuż linii 2-5
$\mathbf{t}^1_{jk} = -\mathbf{t}^2_{ki}$

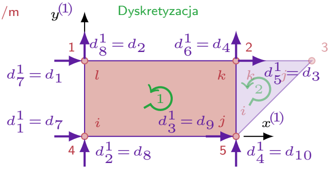




Wektor $\mathbf{p}_\mathrm{b}$ – Element 1

$$\mathbf{p}_\mathrm{b}^1 = \int_{\Gamma^1_{kl}} {\mathbf{N}^1}^\mathrm{T}\mathbf{t}\mathrm{d}\Gamma + \int_{\Gamma^1_{li}} {\mathbf{N}^1}^\mathrm{T}\mathbf{t}\mathrm{d}\Gamma$$





Przykład

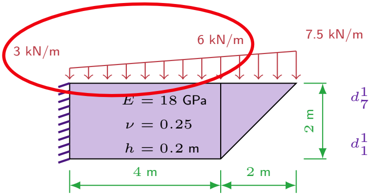
Statyka tarczy

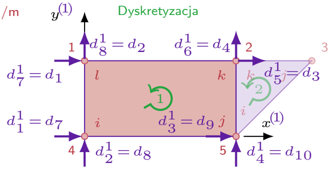
Wektor $\mathbf{p}_\mathrm{b}$ – Element 1

$$\mathbf{p}_\mathrm{b}^1 = \int_{\Gamma^1_{kl}} {\mathbf{N}^1}^\mathrm{T} \mathbf{t} \mathrm{d}\Gamma + \int_{\Gamma^1_{li}} {\mathbf{N}^1}^\mathrm{T} \mathbf{t} \mathrm{d}\Gamma$$

$$\int_{\Gamma^1_{kl}} {\mathbf{N}^1}^\mathrm{T} \mathbf{t} \mathrm{d}\Gamma = \int_0^4 \left(\mathbf{N}^1(x^{(1)}, y^{(1)} = 2)\right)^\mathrm{T} \begin{bmatrix} 0 \\ -3\left(1 - \frac{x^{(1)}}{4}\right) - 6\frac{x^{(1)}}{4} \end{bmatrix} \mathrm{d}x^{(1)}$$

$$= \{0\ 0\ 0\ 0\ 0\ \text{-}10\ 0\ \text{-}8\}$$



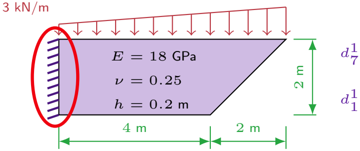


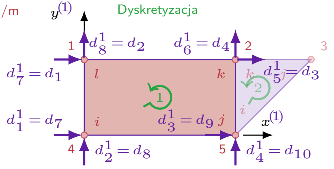
Wektor $\mathbf{p}_\mathrm{b}$ – Element 1

$$\mathbf{p}_\mathrm{b}^1 = \{0\ 0\ 0\ 0\ 0\ \text{-}10\ 0\ \text{-}8\} + \int_{\Gamma_{li}^1} \mathbf{N}^{1\,\mathrm{T}} \mathbf{t}\,\mathrm{d}\Gamma$$

$$\int_{\Gamma_{li}^1} \mathbf{N}^{1\,\mathrm{T}} \mathbf{t}\,\mathrm{d}\Gamma = \int_0^2 \left(\mathbf{N}^1(x^{(1)} = 0, y^{(1)})\right)^\mathrm{T} \mathbf{t}\,\mathrm{d}y^{(1)}$$

$$= \{R_1^1\ R_2^1\ 0\ 0\ 0\ 0\ R_7^1\ R_8^1\}$$

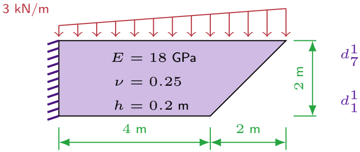


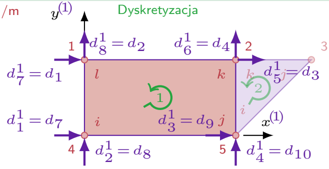


Wektor $\mathbf{p}_\mathrm{b}$ – Element 1

$$\mathbf{p}^1_\mathrm{b} = \{0\ 0\ 0\ 0\ 0\ \text{-}10\ 0\ \text{-}8\} + \{R^1_1\ R^1_2\ 0\ 0\ 0\ 0\ R^1_7\ R^1_8\}$$

$$\mathbf{p}^1_\mathrm{b} = \begin{bmatrix} R^1_1 \\ R^1_2 \\ 0 \\ 0 \\ 0 \\ \text{-}10 \\ R^1_7 \\ R^1_8\text{-}8 \end{bmatrix}$$



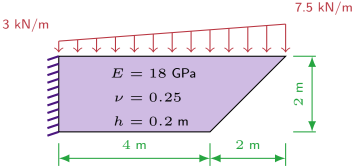


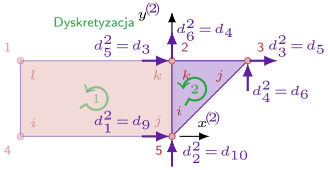
Przykład

Statyka tarczy

Wektor $\mathbf{p}_b$ – Element 2

$$\mathbf{p}_\mathsf{b}^2 = \int_{\Gamma^2_{ij}} \mathbf{N}^{2\,\mathrm{T}} \mathbf{t} \mathrm{d}\Gamma + \int_{\Gamma^2_{jk}} \mathbf{N}^{2\,\mathrm{T}} \mathbf{t} \mathrm{d}\Gamma + \int_{\Gamma^2_{ki}} \mathbf{N}^{2\,\mathrm{T}} \mathbf{t} \mathrm{d}\Gamma$$





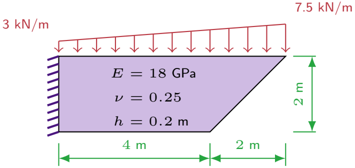
Przykład

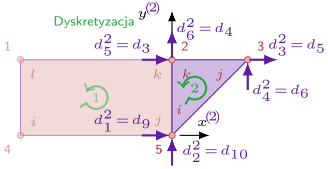
Statyka tarczy

Wektor $\mathbf{p}_b$ – Element 2

$$\mathbf{p}_\mathrm{b}^2 = \int_{\Gamma_{ij}^2} \mathbf{N}^{2^\mathrm{T}} \underbrace{\mathbf{t}}_{w.b.=0} \mathrm{d}\Gamma + \int_{\Gamma_{jk}^2} \mathbf{N}^{2^\mathrm{T}} \mathbf{t}\mathrm{d}\Gamma + \int_{\Gamma_{ki}^2} \mathbf{N}^{2^\mathrm{T}} \mathbf{t}\mathrm{d}\Gamma$$

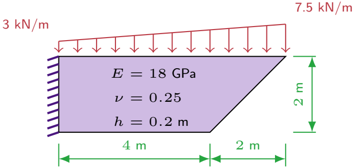
wspólna krawędź
równowaga sił
wzdłuż linii 2-5
$\mathbf{t}_{jk}^1 = -\mathbf{t}_{ki}^2$

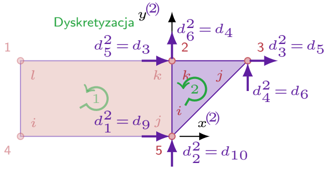




Wektor $\mathbf{p}_b$ – Element 2

$$\mathbf{p}^2_{\mathsf{b}} = -\int_{\Gamma^2_{jk}} \mathbf{N}^{2^{\mathrm{T}}} \mathbf{t} \mathrm{d}\Gamma$$



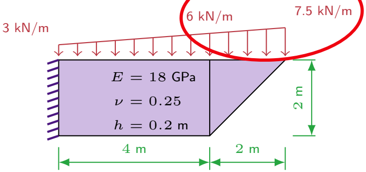


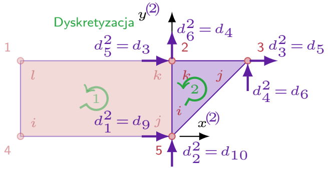
Wektor $\mathbf{p}_b$ – Element 2

$$\mathbf{p}_\mathrm{b}^2 = -\int_{\Gamma_{jk}^2} \mathbf{N}^{2\,\mathrm{T}} \mathbf{t}\,\mathrm{d}\Gamma$$

$$\int_{\Gamma_{jk}^2} \mathbf{N}^{2\,\mathrm{T}} \mathbf{t}\,\mathrm{d}\Gamma = \int_0^2 \left(\mathbf{N}^1(x^{(2)}, y^{(2)} = 2)\right)^\mathrm{T} \left[\begin{array}{c} 0 \\ -6\left(1 - \frac{x^{(2)}}{2}\right) - 7.5\,\frac{x^{(2)}}{2} \end{array}\right] \mathrm{d}x^{(2)}$$

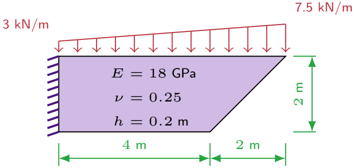
$$= \{0\ 0\ 0\ \text{-}7\ 0\ \text{-}6.5\}$$

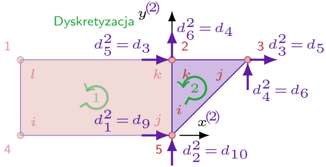




Wektor $\mathbf{p}_b$ – Element 2

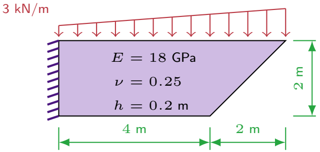
$$\mathbf{p}_\mathsf{b}^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -7 \\ 0 \\ -6.5 \end{bmatrix}$$

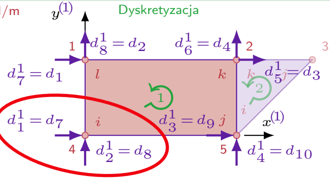




Agregacja – Macierze Boole'a $\mathbb{B}$

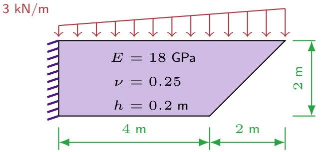
$$
\begin{array}{rc l}
 & & \text{loc. no.} \\
\mathbb{B}^1 = & \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} \\
\text{glob. no.} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{matrix} &
\end{array}
$$

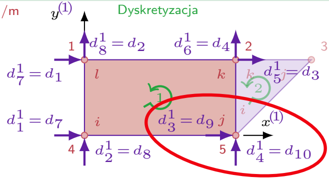




Agregacja – Macierze Boole'a $\mathbb{B}$

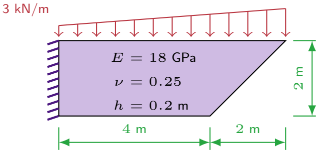
$$
\mathbb{B}^1 = \begin{array}{r|c}
& \text{loc. no.} \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} \\
\text{glob. no.} \quad \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{matrix} &
\end{array}
$$

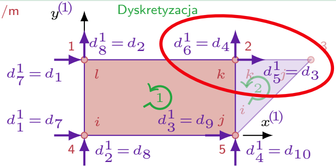




Agregacja – Macierze Boole'a $\mathbb{B}$

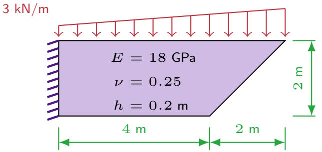
$$
\begin{array}{c c}
 & \text{loc. no.} \\
\mathbb{B}^1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \\
\text{glob. no.} \quad \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} &
\end{array}
$$

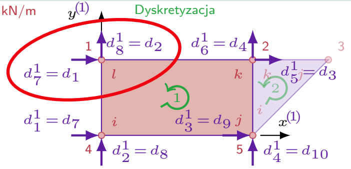




Agregacja – Macierze Boole'a $\mathbb{B}$

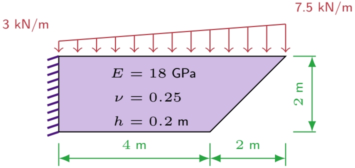
$$
\begin{array}{rcc}
 & & \text{loc. no.} \\
\mathbb{B}^1 = & \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} \\
\text{glob. no.} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{matrix} &
\end{array}
$$

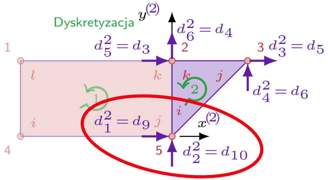




Agregacja – Macierze Boole'a $\mathbb{B}$

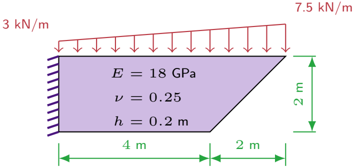
$$
\begin{array}{c|c}
 & \text{loc. no.} \\
\mathbb{B}^2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \\
\text{glob. no.} \quad \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{matrix} &
\end{array}
$$

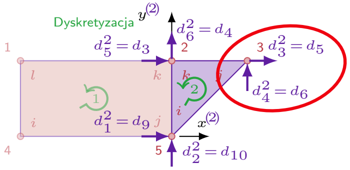




Agregacja – Macierze Boole'a $\mathbb{B}$

$$
\begin{array}{c}
\mathbb{B}^2 = \left[\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\begin{array}{c} \text{loc. no.} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\\
\text{glob. no.} \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10
\end{array}
$$



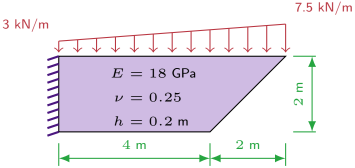


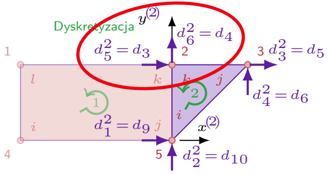
Przykład

Statyka tarczy

Agregacja – Macierze Boole'a $\mathbb{B}$

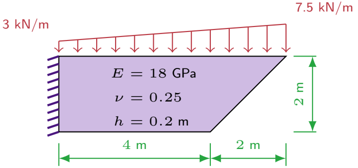
$$
\begin{array}{c|c}
 & \text{loc. no.} \\
\mathbb{B}^2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \\
\text{glob. no.} \quad \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{matrix} &
\end{array}
$$

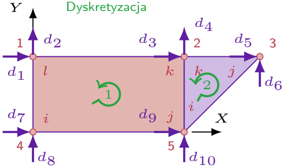




Agregacja - Macierz sztywności

$$\mathbf{K} = \mathbb{B}^{1\,\mathrm{T}} \mathbf{K}^1 \mathbb{B}^1 + \mathbb{B}^{2\,\mathrm{T}} \mathbf{K}^2 \mathbb{B}^2$$

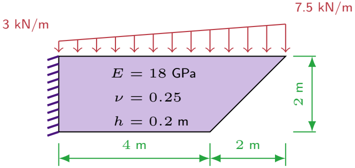


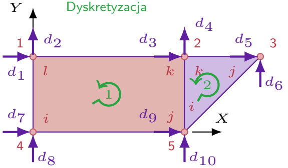


Agregacja - Macierz sztywności

$$\mathbf{K} = {\mathbb{B}^1}^{\mathrm{T}} \mathbf{K}^1 \, \mathbb{B}^1 + {\mathbb{B}^2}^{\mathrm{T}} \mathbf{K}^2 \, \mathbb{B}^2$$

$$\mathbf{K} = \begin{bmatrix} 16 & -6 & -1.6 & 1.2 & 0 & 0 & -6.4 & -1.2 & -8 & 6 \\ -6 & 28 & -1.2 & 10.4 & 0 & 0 & 1.2 & -24.4 & 6 & -14 \\ -1.6 & -1.2 & 42.4 & -6 & -19.2 & 7.2 & -8 & -6 & -13.6 & 6 \\ 1.2 & 10.4 & -6 & 54.4 & 4.8 & -7.2 & -6 & -14 & 6 & -43.6 \\ 0 & 0 & -19.2 & 4.8 & 19.2 & 0 & 0 & 0 & 0 & -4.8 \\ 0 & 0 & 7.2 & -7.2 & 0 & 7.2 & 0 & 0 & -7.2 & 0 \\ -6.4 & 1.2 & -8 & -6 & 0 & 0 & 16 & 6 & -1.6 & -1.2 \\ -1.2 & -24.4 & -6 & -14 & 0 & 0 & 6 & 28 & 1.2 & 10.4 \\ -8 & 6 & -13.6 & 6 & 0 & -7.2 & -1.6 & 1.2 & 23.2 & -6 \\ 6 & -14 & 6 & -43.6 & -4.8 & 0 & -1.2 & 10.4 & -6 & 47.2 \end{bmatrix} \cdot 10^5$$



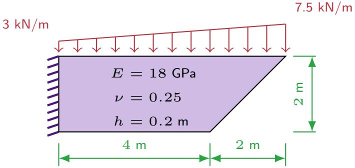


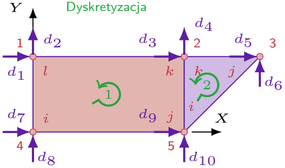
Przykład

Statyka tarczy

Agregacja - Wektor obciążenia

$$\mathbf{p}_{\mathsf{b}} = \mathbb{B}^{1^{\mathrm{T}}} \mathbf{p}_{\mathsf{b}}^{1} + \mathbb{B}^{2^{\mathrm{T}}} \mathbf{p}_{\mathsf{b}}^{2}, \qquad \mathbf{p} = \mathbf{0}$$

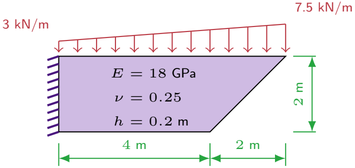


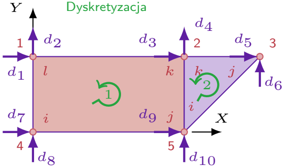


Agregacja - Wektor obciążenia

$$\mathbf{p}_{\mathsf{b}} = \mathbb{B}^{1^{\mathrm{T}}} \mathbf{p}_{\mathsf{b}}^{1} + \mathbb{B}^{2^{\mathrm{T}}} \mathbf{p}_{\mathsf{b}}^{2}, \qquad \mathbf{p} = \mathbf{0}$$

$$\mathbf{p}_{\mathsf{b}} = \begin{bmatrix} 0 \\ -8 \\ 0 \\ -16.5 \\ 0 \\ -7 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} R_7^1 = R_1 \\ R_8^1 = R_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ R_1^1 = R_7 \\ R_2^1 = R_8 \\ 0 \\ 0 \end{bmatrix}$$



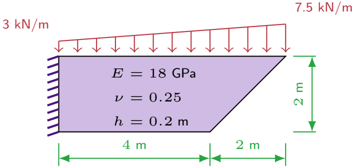


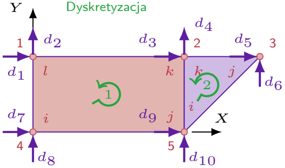
Przykład

Statyka tarczy

Układ równań MES: $\mathbf{K}\mathbf{d} = \mathbf{p} + \mathbf{p}_b$

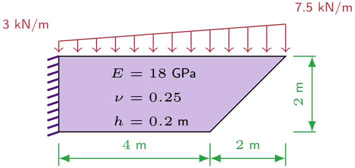
$$
\begin{bmatrix}
16 & -6 & -1.6 & 1.2 & 0 & 0 & -6.4 & -1.2 & -8 & 6 \\
-6 & 28 & -1.2 & 10.4 & 0 & 0 & 1.2 & -24.4 & 6 & -14 \\
-1.6 & -1.2 & 42.4 & -6 & -19.2 & 7.2 & -8 & -6 & -13.6 & 6 \\
1.2 & 10.4 & -6 & 54.4 & 4.8 & -7.2 & -6 & -14 & 6 & -43.6 \\
0 & 0 & -19.2 & 4.8 & 19.2 & 0 & 0 & 0 & 0 & -4.8 \\
0 & 0 & 7.2 & -7.2 & 0 & 7.2 & 0 & 0 & -7.2 & 0 \\
-6.4 & 1.2 & -8 & -6 & 0 & 0 & 16 & 6 & -1.6 & -1.2 \\
-1.2 & -24.4 & -6 & -14 & 0 & 0 & 6 & 28 & 1.2 & 10.4 \\
-8 & 6 & -13.6 & 6 & 0 & -7.2 & -1.6 & 1.2 & 23.2 & -6 \\
6 & -14 & 6 & -43.6 & -4.8 & 0 & -1.2 & 10.4 & -6 & 47.2
\end{bmatrix} \cdot 10^5
\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \\ d_9 \\ d_{10} \end{bmatrix}
=
\begin{bmatrix} 0 \\ -8 \\ 0 \\ -16.5 \\ 0 \\ -7 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
+
\begin{bmatrix} R_1 \\ R_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ R_7 \\ R_8 \\ 0 \\ 0 \end{bmatrix}
$$

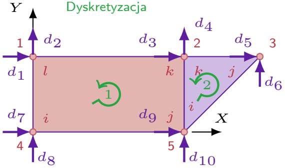




Układ równań MES: $\mathbf{Kd} = \mathbf{p} + \mathbf{p}_b$

$$
\begin{bmatrix}
16 & -6 & -1.6 & 1.2 & 0 & 0 & -6.4 & -1.2 & -8 & 6 \\
-6 & 28 & -1.2 & 10.4 & 0 & 0 & 1.2 & -24.4 & 6 & -14 \\
-1.6 & -1.2 & 42.4 & -6 & -19.2 & 7.2 & -8 & -6 & -13.6 & 6 \\
1.2 & 10.4 & -6 & 54.4 & 4.8 & -7.2 & -6 & -14 & 6 & -43.6 \\
0 & 0 & -19.2 & 4.8 & 19.2 & 0 & 0 & 0 & 0 & -4.8 \\
0 & 0 & 7.2 & -7.2 & 0 & 7.2 & 0 & 0 & -7.2 & 0 \\
-6.4 & 1.2 & -8 & -6 & 0 & 0 & 16 & 6 & -1.6 & -1.2 \\
-1.2 & -24.4 & -6 & -14 & 0 & 0 & 6 & 28 & 1.2 & 10.4 \\
-8 & 6 & -13.6 & 6 & 0 & -7.2 & -1.6 & 1.2 & 23.2 & -6 \\
6 & -14 & 6 & -43.6 & -4.8 & 0 & -1.2 & 10.4 & -6 & 47.2
\end{bmatrix} \cdot 10^5
\begin{bmatrix} 0 \\ 0 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ 0 \\ 0 \\ d_9 \\ d_{10} \end{bmatrix}
=
\begin{bmatrix} 0 \\ -8 \\ 0 \\ -16.5 \\ 0 \\ -7 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
+
\begin{bmatrix} R_1 \\ R_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ R_7 \\ R_8 \\ 0 \\ 0 \end{bmatrix}
$$





Przykład

Statyka tarczy

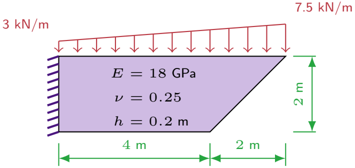
Układ równań MES: $\mathbf{Kd} = \mathbf{p} + \mathbf{p}_b$

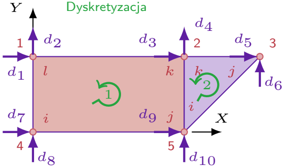
$$
\begin{bmatrix}
16 & -6 & -1.6 & 1.2 & 0 & 0 & -6.4 & -1.2 & -8 & 6 \\
-6 & 28 & -1.2 & 10.4 & 0 & 0 & 1.2 & -24.4 & 6 & -14 \\
-1.6 & -1.2 & 42.4 & -6 & -19.2 & 7.2 & -8 & -6 & -13.6 & 6 \\
1.2 & 10.4 & -6 & 54.4 & 4.8 & -7.2 & -6 & -14 & 6 & -43.6 \\
0 & 0 & -19.2 & 4.8 & 19.2 & 0 & 0 & 0 & 0 & -4.8 \\
0 & 0 & 7.2 & -7.2 & 0 & 7.2 & 0 & 0 & -7.2 & 0 \\
-6.4 & 1.2 & -8 & -6 & 0 & 0 & 16 & 6 & -1.6 & -1.2 \\
-1.2 & -24.4 & -6 & -14 & 0 & 0 & 6 & 28 & 1.2 & 10.4 \\
-8 & 6 & -13.6 & 6 & 0 & -7.2 & -1.6 & 1.2 & 23.2 & -6 \\
6 & -14 & 6 & -43.6 & -4.8 & 0 & -1.2 & 10.4 & -6 & 47.2
\end{bmatrix} \cdot 10^5
\begin{bmatrix} 0 \\ 0 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ 0 \\ 0 \\ d_9 \\ d_{10} \end{bmatrix}
=
\begin{bmatrix} 0 \\ -8 \\ 0 \\ -16.5 \\ 0 \\ -7 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
+
\begin{bmatrix} R_1 \\ R_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ R_7 \\ R_8 \\ 0 \\ 0 \end{bmatrix}
$$

Rozwiązanie:

$$\mathbf{d} = \{0\ 0\ 3.881\ \text{-}11.03\ 3.949\ \text{-}19.62\ 0\ 0\ \text{-}3.744\ \text{-}10.75\} \cdot 10^{-5}\ \text{m}$$

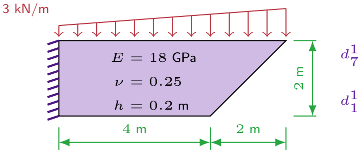
$$\mathbf{R} = \{\text{-}54\ 16.744\ 0\ 0\ 0\ 0\ 54\ 14.756\ 0\ 0\}\ \text{kN}$$

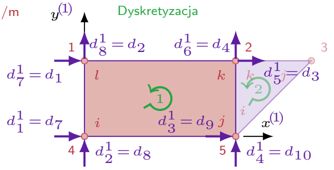




Powrót do elementu: Element 1

$$\mathbf{d}^1 = \mathbb{B}^1 \mathbf{d} = \{0\ 0\ \text{-}3.744\ \text{-}10.75\ 3.881\ \text{-}11.03\ 0\ 0\} \cdot 10^{-5}$$



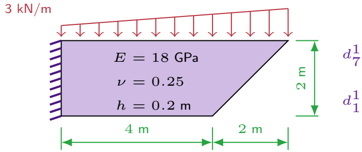


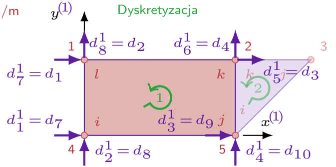
Powrót do elementu: Element 1

$$\mathbf{d}^1 = \mathbb{B}^1\mathbf{d} = \{0\ 0\ \text{-}3.744\ \text{-}10.75\ 3.881\ \text{-}11.03\ 0\ 0\} \cdot 10^{-5}$$

$$\boldsymbol{\varepsilon}^1 = \mathbf{B}^1\mathbf{d}^1$$

$$\boldsymbol{\varepsilon}^1 = \begin{bmatrix} 0.953y - 0.936 \\ -0.034x \\ 0.953x - 0.034y - 2.688 \end{bmatrix} \cdot 10^{-5}, \qquad \boldsymbol{\varepsilon}^1(2,1) = \begin{bmatrix} 1.708 \\ 6.831 \\ -81.600 \end{bmatrix} \cdot 10^{-7}$$





Powrót do elementu: Element 1

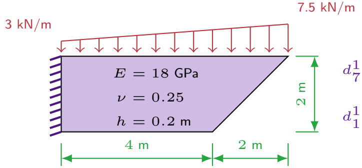
$$\mathbf{d}^1 = \mathbb{B}^1\mathbf{d} = \{0\ 0\ \text{-}3.744\ \text{-}10.75\ 3.881\ \text{-}11.03\ 0\ 0\} \cdot 10^{-5}$$

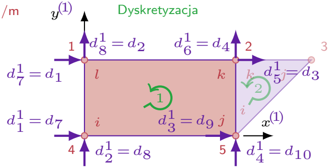
$$\boldsymbol{\varepsilon}^1 = \mathbf{B}^1\mathbf{d}^1$$

$$\boldsymbol{\varepsilon}^1 = \begin{bmatrix} 0.953y - 0.936 \\ -0.034x \\ 0.953x - 0.034y - 2.688 \end{bmatrix} \cdot 10^{-5}, \qquad \boldsymbol{\varepsilon}^1(2,1) = \begin{bmatrix} 1.708 \\ 6.831 \\ -81.600 \end{bmatrix} \cdot 10^{-7}$$

$$\boldsymbol{\sigma}^1 = \mathbf{D}\boldsymbol{\varepsilon}^1$$

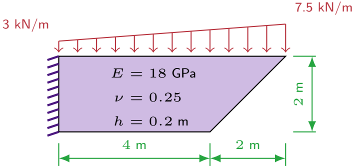
$$\boldsymbol{\sigma}^1 = \begin{bmatrix} 182.976y - 179.712 - 1.632x \\ 45.744y - 44.928 - 6.528x \\ 68.616x - 2.448y - 193.536 \end{bmatrix}, \qquad \boldsymbol{\sigma}^1(2,1) = \begin{bmatrix} 0 \\ -12.297 \\ -58.750 \end{bmatrix} \text{ kPa}$$

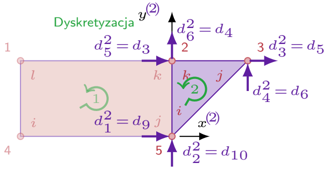




Powrót do elementu: Element 2

$$\mathbf{d}^2 = \mathbb{B}^2 \mathbf{d} = \{\text{-}3.744\ \text{-}10.75\ 3.949\ \text{-}19.62\ 3.881\ \text{-}11.03\} \cdot 10^{-5}$$





Przykład

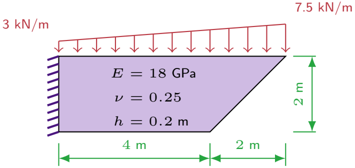
Statyka tarczy

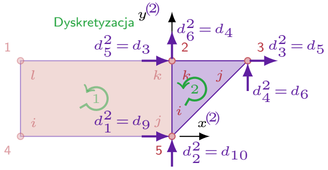
Powrót do elementu: Element 2

$$\mathbf{d}^2 = \mathbb{B}^2\mathbf{d} = \{\text{-3.744 -10.75 3.949 -19.62 3.881 -11.03}\} \cdot 10^{-5}$$

$$\boldsymbol{\varepsilon}^2 = \mathbf{B}^2\mathbf{d}^2$$

$$\boldsymbol{\varepsilon}^2 = \begin{bmatrix} 3.416 \\ 13.660 \\ -48.610 \end{bmatrix} \cdot 10^{-7}$$





Powrót do elementu: Element 2

$$\mathbf{d}^2 = \mathbb{B}^2\mathbf{d} = \{\text{-3.744 -10.75 3.949 -19.62 3.881 -11.03}\} \cdot 10^{-5}$$

$$\boldsymbol{\varepsilon}^2 = \mathbf{B}^2\mathbf{d}^2$$

$$\boldsymbol{\varepsilon}^2 = \begin{bmatrix} 3.416 \\ 13.660 \\ -48.610 \end{bmatrix} \cdot 10^{-7}$$

$$\boldsymbol{\sigma}^2 = \mathbf{D}\boldsymbol{\varepsilon}^2$$

$$\boldsymbol{\sigma}^2 = \begin{bmatrix} 0 \\ -24.593 \\ -35.000 \end{bmatrix} \text{ kPa}$$