Computer methods in Civil Engineering
Instruction for **project no 3** – *FEM in 1D non-linear problems*
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1. Problem formulation

The following non-linear differential equation is given:

$$\frac{d^2y}{dx^2}+y^2\cdot\frac{dy}{dx}=f,\quad x\in\left[a\quad b\right],\quad y\left(a\right)=y_a,\quad y\left(b\right)=y_b \tag{1.1}$$

In order to apply the FE model, it is required to derive the appropriate variational principle

$$-\int_a^b\frac{dv}{dx}\frac{dy}{dx}dx+\left[v\frac{dy}{dx}\right]_a^b+\int_a^b v\cdot y^2\cdot\frac{dy}{dx}dx=\int_a^b v\cdot f\cdot dx \tag{1.2}$$

Assuming test function v in such a manner, that it vanishes at boundary, $v\left(a\right)=v\left(b\right)=0$, the final variational form is

$$-\int_a^b\frac{dv}{dx}\frac{dy}{dx}dx+\int_a^b v\cdot y^2\cdot\frac{dy}{dx}dx=\int_a^b v\cdot f\cdot dx \tag{1.3}$$

It is non-linear though, with the respect to the second component on the left-hand side.

2. FE approximation

Linearization of (1.3) shall be performed following the FE approximation, using linear shape functions. Interval [a, b] is discretized with N finite elements of equal length each $h=\dfrac{b-a}{N}$ which gives the total number of nodes $n=N+1$ with coordinates $\boldsymbol{X}=\left\{X_i\ ,i=1,2,...,n\right\}$ (and degrees of freedom Y_i). On the level of the i-th finite element $x^e\in\left[x_1=X_i\quad x_2=X_{i+1}\right]$, the following interpolation of both trial y and test v functions is provided

$$\begin{aligned}y\left(x\right)&=\left[N_1^e\left(x\right)\quad N_2^e\left(x\right)\right]\begin{bmatrix}Y_1^e\\Y_2^e\end{bmatrix}=\boldsymbol{N}^e\left(x\right)\cdot\boldsymbol{Y}^e\\v\left(x\right)&=\left[N_1^e\left(x\right)\quad N_2^e\left(x\right)\right]\begin{bmatrix}V_1^e\\V_2^e\end{bmatrix}=\boldsymbol{N}^e\left(x\right)\cdot\boldsymbol{V}^e=\left(\boldsymbol{V}^e\right)^T\cdot\left(\boldsymbol{N}^e\left(x\right)\right)^T\end{aligned} \tag{2.1}$$

In the similar manner, their derivatives are expressed

$$\frac{d}{dx}y(x)=\left[\frac{d}{dx}N_1^e(x)\quad \frac{d}{dx}N_2^e(x)\right]\begin{bmatrix}Y_1^e\\ Y_2^e\end{bmatrix}=\frac{d}{dx}\boldsymbol{N}^e(x)\cdot\boldsymbol{Y}^e=\boldsymbol{B}^e(x)\cdot\boldsymbol{Y}^e$$
$$\frac{d}{dx}v(x)=\left[\frac{d}{dx}N_1^e(x)\quad \frac{d}{dx}N_2^e(x)\right]\begin{bmatrix}V_1^e\\ V_2^e\end{bmatrix}=\frac{d}{dx}\boldsymbol{N}^e(x)\cdot\boldsymbol{V}^e=\left(\boldsymbol{V}^e\right)^T\cdot\left(\boldsymbol{B}^e(x)\right)^T \tag{2.2}$$

with

$$N_1^e(x)=\frac{x-x_2}{x_1-x_2}\quad,\quad N_2^e(x)=\frac{x-x_1}{x_2-x_1}$$
$$B_1^e(x)=\frac{1}{x_1-x_2}\quad,\quad B_2^e(x)=\frac{1}{x_2-x_1} \tag{2.3}$$

Above formulas should be substituted to (1.3). Using the arbitrary selection of function v, we obtain the final FE formulation (variable (x) has been omitted)

$$-\int_{x_1}^{x_2}\left(\boldsymbol{B}^e\right)^T\boldsymbol{B}^e\boldsymbol{Y}^e dx+\int_{x_1}^{x_2}\left(\boldsymbol{N}^e\right)^T\left(\boldsymbol{N}^e\boldsymbol{Y}^e\right)^2\boldsymbol{B}^e\boldsymbol{Y}^e dx=\int_{x_1}^{x_2}\left(\boldsymbol{N}^e\right)^T f\,dx \tag{2.4}$$

or in a brief notation

$$\boldsymbol{R}^e\left(\boldsymbol{Y}^e\right)=\boldsymbol{0} \tag{2.5}$$

where $\boldsymbol{R}^e\left(\boldsymbol{Y}^e\right)$ is the residuum of equation (2.4) and it is defined as

$$\boldsymbol{R}^e\left(\boldsymbol{Y}^e\right)=-\int_{x_1}^{x_2}\left(\boldsymbol{B}^e\right)^T\boldsymbol{B}^e\boldsymbol{Y}^e dx+\int_{x_1}^{x_2}\left(\boldsymbol{N}^e\right)^T\left(\boldsymbol{N}^e\boldsymbol{Y}^e\right)^2\boldsymbol{B}^e\boldsymbol{Y}^e dx-\int_{x_1}^{x_2}\left(\boldsymbol{N}^e\right)^T\frac{j}{M}f\,dx \tag{2.6}$$

In the case the load is "portioned" (divided into M increments), in the above formula we take j-th load increment $\frac{j}{M}f$.

3. Linearization by Newton-Raphson

Linearization of the above equation requires the application of the Newton-Raphson procedure

$$\boldsymbol{K}_T^e\Delta\boldsymbol{Y}^e=-\boldsymbol{R}^e \tag{3.1}$$

where the tangent stiffness matrix $\boldsymbol{K}_T^e$ appears

$$\boldsymbol{K}_T^e=\frac{\partial\boldsymbol{R}^e}{\partial\boldsymbol{Y}^e}=-\int_{x_1}^{x_2}\left(\boldsymbol{B}^e\right)^T\boldsymbol{B}^e dx+\int_{x_1}^{x_2}\left(\boldsymbol{N}^e\right)^T\left(2\left(\boldsymbol{N}^e\boldsymbol{Y}^e\right)\boldsymbol{N}^e\left(\boldsymbol{B}^e\boldsymbol{Y}^e\right)+\left(\boldsymbol{N}^e\boldsymbol{Y}^e\right)^2\boldsymbol{B}^e\right)dx \tag{3.2}$$

After determination of tangent stiffness matrix (3.2) for each element as well as the element residual vector (2.6), one needs to assembly them into the global system

$$\left(\boldsymbol{K}_T^e\right)^{(k)}\left(\Delta\boldsymbol{Y}^e\right)^{(k+1)}=\left(-\boldsymbol{R}^e\right)^{(k)} \tag{3.3}$$

in which the vector of solution increments $\Delta\boldsymbol{Y}^e$ in unknown on the next k+1 iteration step. After its determination, the solution should be updated

$$\left(\boldsymbol{Y}^e\right)^{(k+1)}=\left(\boldsymbol{Y}^e\right)^{(k)}+\left(\Delta\boldsymbol{Y}^e\right)^{(k+1)} \tag{3.4}$$

We proceed as long as the estimated error becomes smaller than the assumed admissible error

$$\varepsilon^{(k+1)}=\frac{\left\|\left(\Delta\boldsymbol{Y}^e\right)^{(k+1)}\right\|}{\left\|\left(\boldsymbol{Y}^e\right)^{(k+1)}\right\|}<\varepsilon_{dop} \tag{3.5}$$

Additionally, one may assume the maximum number of iterations $k_{\max}$.

4. **Initial solution**

We need to assume the initial solution in order to determine first solution increment, by means of tangent stiffness matrix (3.2) and residual vector (2.6). This initial solution has a significant influence on the convergence rate of Newton-Raphson procedure, therefore it has to selected in a proper manner. In most cases, the initial solution is very simple function $y^{(0)}(x)$ (e.g., polynomial, trigonometric function), though satisfying the boundary conditions of the original problem (1.1). On its basis, one determines the initial solution at nodes $\boldsymbol{Y}^{(0)}$, namely values of $y^{(0)}(x)$ for $x=\boldsymbol{X}$. In case of the considered problem, we assume the initial solution as the linear function

$$y^{(0)}(x)=Ax+B \tag{3.6}$$

where coefficients A and B have to be determined on the basis of boundary conditions

$$\begin{cases} y^{(0)}(a)=Aa+B=y_a \\ y^{(0)}(b)=Ab+B=y_b \end{cases} \tag{3.7}$$

hence

$$A=\frac{y_b-y_a}{b-a},\quad B=y_a-Aa \tag{3.8}$$

5. **Scope of the project**

Incomplete Matlab code may be downloaded from http://www.L5.pk.edu.pl/~slawek/Comp_Meths_2019/proj3.m

One needs to assume the beginning of the interval (a) and its end (b), maximum number of iterations $k_{\max}$ (e.g., 10), admissible solution error (e.g., 10^{-6}), number of finite elements N and number of load increments M (for the very first time, please assume 1 increment).

The exact solution that satisfies the initial differential equation should be assumed as a quadratic polynomial (parabola). On its basis, the right side of the differential equation (function f) should be calculated. Moreover, boundary values y_a and y_b should be computed (in Matlab), based on this solution. Subsequent commands in the file should be completed according to the comments and references to the formulas from the instructions.

After finishing your work, run the program. In case of convergence problems (when the FE solution does not resemble a strict solution), please try:

- reduce the length of the interval,
- increase the number of elements,
- increase the number of load increments (gradually, at 2, then at 3).

Further possible paths of proceedings (for those who want to have a higher grade from the project):

- replacement of the current version of the N-R method with the initial or modified version,
- application of hierarchical interpolation with an additional square shape function,
- replacement of the right boundary condition from essential to natural,
- plot of the solution convergence graph for the selected node.