Weighted residual method

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Weighted residual method

Strong formulation - local model

$$A(x)y''(x) + B(x)y'(x) + C(x)y(x) = D(x)\ , \quad x \in (x_a, x_b)$$

+ (example) boundary conditions

- $y(x_a) = \widehat{a}$ – essential (Dirichlet) boundary condition
- $y'(x_b) = \widehat{b}$ – natural (Neumann) boundary condition

Function approximation

$$\tilde{y} = \phi_0 + \sum_{i=1}^{n} \phi_i c_i = \phi_0 + \boldsymbol{\phi c}$$

ϕ_i – (known) basis functions, c_i – (unknown) coefficients

Residuum

$$R(x) = A(x)\tilde{y}''(x) + B(x)\tilde{y}'(x) + C(x)\tilde{y}(x) - D(x)$$

Weighted residual method

Residuum minimization

$$\int_{x_a}^{x_b} w(x)R(x)\mathrm{d}x = 0 \quad \forall w$$

- $w(x) \neq 0$ – weighting function

$$\int_{x_a}^{x_b} w(x)\left(A(x)\tilde{y}''(x) + B(x)\tilde{y}'(x) + C(x)\tilde{y}(x) - D(x)\right)\mathrm{d}x = 0$$

Different WRM variants depending on weighting function

- collocation method – $w_i = \delta(x - x_i)$
- least squares method – $w_i = \dfrac{\mathrm{d}R}{\mathrm{d}c_i}$
- Bubnov-Galerkin method – $w_i = \phi_i$

Weighted residual method

Weak formulation - global model ($\forall w \neq 0$)

$$\int_{x_a}^{x_b} w(x)\left(A(x)\tilde{y}''(x) + B(x)\tilde{y}'(x) + C(x)\tilde{y}(x) - D(x)\right)\mathrm{d}x = 0$$

$$\int_{x_a}^{x_b} w(x)A(x)\tilde{y}''(x)\mathrm{d}x +$$

$$+\int_{x_a}^{x_b} w(x)B(x)\tilde{y}'(x)\mathrm{d}x + \int_{x_a}^{x_b} w(x)C(x)\tilde{y}(x)\mathrm{d}x - \int_{x_a}^{x_b} w(x)D(x)\mathrm{d}x = 0$$

$$w(x)A(x)\tilde{y}'(x)\Big|_{x_a}^{x_b} - \int_{x_a}^{x_b} w'(x)A(x)\tilde{y}'(x)\mathrm{d}x - \int_{x_a}^{x_b} w(x)A'(x)\tilde{y}'(x)\mathrm{d}x +$$

$$+\int_{x_a}^{x_b} w(x)B(x)\tilde{y}'(x)\mathrm{d}x + \int_{x_a}^{x_b} w(x)C(x)\tilde{y}(x)\mathrm{d}x - \int_{x_a}^{x_b} w(x)D(x)\mathrm{d}x = 0$$

Weighted residual method

Weak formulation - global model ($\forall w \neq 0$)

$$w(x)A(x)\tilde{y}'(x)\Big|_{x_a}^{x_b} - \int_{x_a}^{x_b} w'(x)A(x)\tilde{y}'(x)\mathrm{d}x - \int_{x_a}^{x_b} w(x)A'(x)\tilde{y}'(x)\mathrm{d}x +$$

$$+ \int_{x_a}^{x_b} w(x)B(x)\tilde{y}'(x)\mathrm{d}x + \int_{x_a}^{x_b} w(x)C(x)\tilde{y}(x)\mathrm{d}x - \int_{x_a}^{x_b} w(x)D(x)\mathrm{d}x = 0$$

$$w(x_b)A(x_b)\ \tilde{y}'(x_b)\ \overset{\widehat{b}}{\boxed{\tilde{y}'(x_b)}}\ \widehat{b} - w(x_a)A(x_a)\tilde{y}'(x_a) - \int_{x_a}^{x_b} w'(x)A(x)\tilde{y}'(x)\mathrm{d}x +$$

$$+ \int_{x_a}^{x_b} w(x)[B(x) - A'(x)]\tilde{y}'(x)\mathrm{d}x + \int_{x_a}^{x_b} w(x)C(x)\tilde{y}(x)\mathrm{d}x - \int_{x_a}^{x_b} w(x)D(x)\mathrm{d}x = 0$$

+ essential boundary condition

- $y(x_a) = \widehat{a}$

Example formulation - one-dimensional heat flow

Heat amount

$$Q \qquad [\mathrm{J}]$$

amount of heat energy

Heat flux

$$H = \frac{\partial Q}{\partial t} \qquad [\mathrm{J/s{=}W}]$$

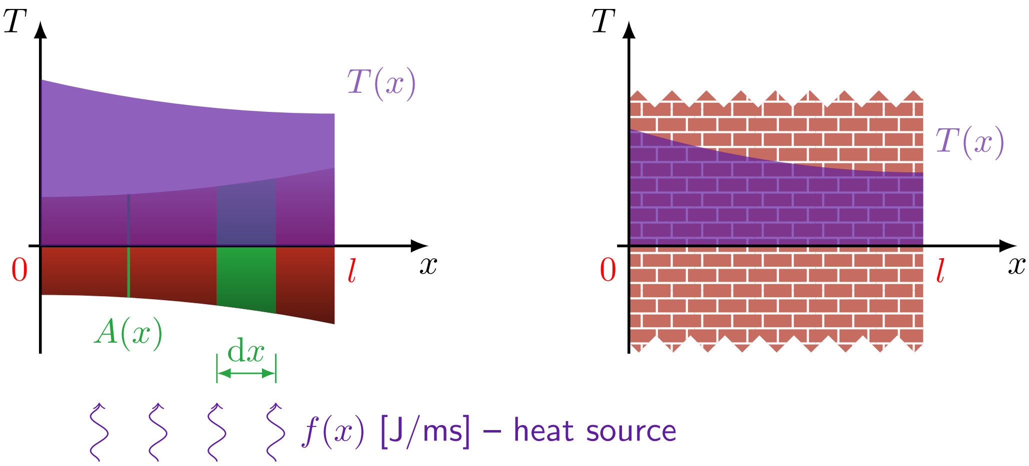
amount of heat per unit time

Heat flux density

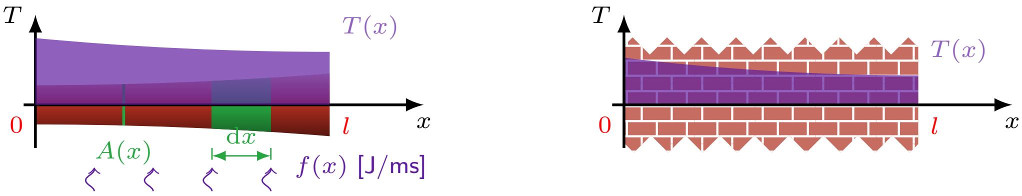
$$q_{\mathrm{n}} = \frac{\partial H}{\partial A} \qquad [\mathrm{W/m^2}] \qquad \text{for 1D: } H(x) = A(x)q_x(x)$$

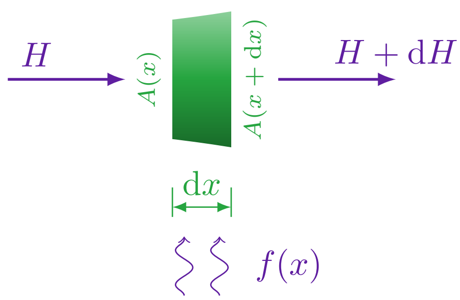
heat flux per unit area

Example formulation - one-dimensional heat flow



Example formulation - one-dimensional heat flow





$$H + f\,\mathrm{d}x = H + \mathrm{d}H \Rightarrow \frac{\mathrm{d}H}{\mathrm{d}x} = f$$

Fourier's law ($H = Aq_x$)

$$q_x = -k\frac{\mathrm{d}T}{\mathrm{d}x}$$

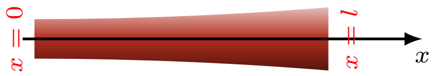
k – heat conductivity coefficient

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(Ak\frac{\mathrm{d}T}{\mathrm{d}x}\right) = f$$

for $Ak = \mathrm{const}$

$$Ak\frac{\mathrm{d}^2T}{\mathrm{d}x^2} + f = 0$$

Example formulation - one-dimensional heat flow



Local model (strong formulation)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(Ak\frac{\mathrm{d}T}{\mathrm{d}x}\right) + f = 0$$

+ boundary conditions

$$q(x=0) = -\left(k\frac{\mathrm{d}T}{\mathrm{d}x}\right)\bigg|_{x=0} = \widehat{q} \qquad \text{natural (Neumann) b.c.}$$

$$T(x=l) = \widehat{T} \qquad \text{essential (Dirichlet) b.c.}$$

Example formulation - one-dimensional heat flow

Global model (weak formulation)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(Ak\frac{\mathrm{d}T}{\mathrm{d}x}\right) + f = 0 \quad \Big| \cdot w \Big| \int_0^l$$

$$\int_0^l w\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(Ak\frac{\mathrm{d}T}{\mathrm{d}x}\right) + f\right)\mathrm{d}x = 0, \quad \forall w \neq 0$$

$$\int_0^l w\frac{\mathrm{d}}{\mathrm{d}x}\left(Ak\frac{\mathrm{d}T}{\mathrm{d}x}\right)\mathrm{d}x + \int_0^l wf\mathrm{d}x = 0$$

$$\left[wAk\frac{\mathrm{d}T}{\mathrm{d}x}\right]\Bigg|_0^l - \int_0^l \frac{\mathrm{d}w}{\mathrm{d}x}\left(Ak\frac{\mathrm{d}T}{\mathrm{d}x}\right)\mathrm{d}x + \int_0^l wf\mathrm{d}x = 0$$

$$\int_0^l \frac{\mathrm{d}w}{\mathrm{d}x}\left(Ak\frac{\mathrm{d}T}{\mathrm{d}x}\right)\mathrm{d}x = -(wAq_x)\bigg|_{x=l} + (wAq_x)\bigg|_{x=0} + \int_0^l wf\mathrm{d}x = 0$$

Example formulation - one-dimensional heat flow

Local model (strong formulation)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(Ak\frac{\mathrm{d}T}{\mathrm{d}x}\right)+f=0$$

$$q_x(x=0)=-\left(k\frac{\mathrm{d}T}{\mathrm{d}x}\right)\bigg|_{x=0}=\widehat{q}$$ natural (Neumann) b.c.

$$T(x=l)=\widehat{T}$$ essential (Dirichlet) b.c.

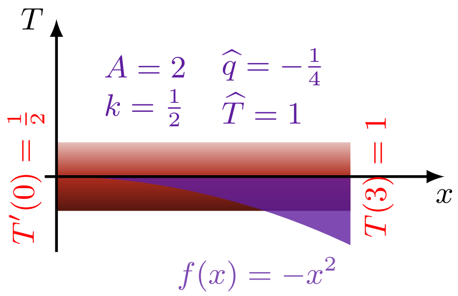
Global model (weak formulation)

$$\int_0^l \frac{\mathrm{d}w}{\mathrm{d}x}\left(Ak\frac{\mathrm{d}T}{\mathrm{d}x}\right)\mathrm{d}x=-(wAq_x)\bigg|_{x=l}+(wA)\bigg|_{x=0}\widehat{q}+\int_0^l wf\mathrm{d}x=0$$

$$T(x=l)=\widehat{T}$$ essential (Dirichlet) b.c.

Example

Derivation of weak format equation for 1D problem



Strong format

$$T''-x^2=0$$

+ boundary conditions

$$q_x(0)=-\frac{1}{2}T'(0)=-\frac{1}{4}$$

$$T(3)=\widehat{T}=1$$

Weak format

$$\int_0^3 w\left[T''-x^2\right]\mathrm{d}x=0\quad\forall w$$

$$-\int_0^3 w'T'\mathrm{d}x+w(3)T'(3)-w(0)T'(0)-\int_0^3 wx^2\mathrm{d}x=0$$

$$\int_0^3 w'T'\mathrm{d}x=w(3)T'(3)-w(0)\cdot\frac{1}{2}-\int_0^3 wx^2\mathrm{d}x$$

+ boundary condition $T(3)=1$