Solution of ODE by FEM

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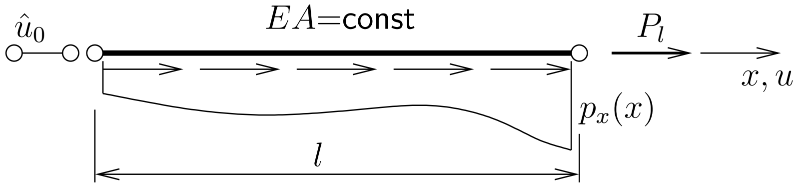
Physical problem - example

Bar under distributed loading – boundary value problem



(1) Balance $\frac{\mathrm{d}N}{\mathrm{d}x} \equiv N' = -p_x$

(2) Kinematic $\epsilon_0 = \frac{\mathrm{d}u}{\mathrm{d}x} \equiv u'$

(3) Constitutive $N = EA\epsilon_0$

Substituting (3)→(2):

(4) Force-displt $N = EAu'$

Substituting (4)→(1):

Local model: $\boxed{EAu'' = -p_x}$

Two boundary conditions:

either essential or natural

At left end $x = 0$ either $u_0 = \hat{u}_0$ or $u'_0 = \frac{P_0}{EA}$

At right end $x = l$ either $u_l = \hat{u}_l$ or $u'_l = \frac{P_l}{EA}$

Well-posed problem – min. one b.c. is essential

B.cs can be homogeneous or non-homogeneous

E.g. $\boxed{u_0 = 0 \text{ and } u'_l = \frac{P_l}{EA}}$

Weighted residual method

For FEM we need a global model. Principle of virtual work or minimum total potential energy are global models. If local model is given, so-called weighted residual method can be used.

Equivalent global model

Rework differential equation into residuum form

$$R(x) = EAu''(x) + p_x(x) = 0$$

We look for approximate solution $\tilde{u}$ for which

$$R(x) = EA\tilde{u}''(x) + p_x(x) \neq 0$$

In weighted residual method we require that

$$\int_0^l w(x)R(x)\mathrm{d}x = 0 \quad \forall w \neq 0$$

Boundary conditions hold

Weighted residual method

Weak (global) formulation

Substitute for residuum

$$\int_0^l w\,(EAu'' + p_x)\mathrm{d}x = 0 \quad \forall w$$

$$\int_0^l w\,EAu''\mathrm{d}x + \int_0^l w\,p_x\mathrm{d}x = 0 \quad \forall w$$

Integrate by parts to reduce continuity requirements

$$-\int_0^l w'\,EAu'\mathrm{d}x + [w\,EAu']_0^l + \int_0^l w\,p_x\mathrm{d}x = 0 \quad \forall w$$

Natural boundary condition introduced into boundary term, essential boundary condition must be imposed.
C^0-continuous approximation sufficient.

Weighted residual method

Virtual work principle

Weak format rewritten

$$\int_0^l w'\, EAu' \mathrm{d}x = [w\, EAu']_0^l + \int_0^l w\, p_x \mathrm{d}x \quad \forall w$$

Weight function interpreted as variation of longitudinal displacement δu

$$\int_0^l \delta u'\, EAu' \mathrm{d}x = [\delta u\, EAu']_0^l + \int_0^l \delta u\, p_x \mathrm{d}x \quad \forall \delta u$$

Rewrite as virtual work principle

$$\int_0^l \delta\epsilon_0\, N \mathrm{d}x = [\delta u\, N]_0^l + \int_0^l \delta u\, p_x \mathrm{d}x\,, \quad \delta W_{\text{int}} = \delta W_{\text{ext}} \quad \forall \delta u$$

Virtual displacement δu is kinematically admissible if it satisfies homogeneous essential boundary conditions

Approximate solution

Bubnov-Galerkin method

Weak formulation of BVP

$$\int_0^l w'\, EAu' \mathrm{d}x = [w\, EAu']_0^l + \int_0^l w\, p_x \mathrm{d}x \quad \forall w \quad \text{plus b.cs}$$

Assume global approximation $\tilde{u}$ as follows

$$\tilde{u} = \phi_0 + \sum_{i=1}^{n} \phi_i c_i = \phi_0 + \boldsymbol{\phi}\, \boldsymbol{c}$$

$\phi_0, \phi_i, i = 1 \dots n$ – (known, linearly independent) basis functions

(ϕ_0 satisfies non-homogeneous essential b.cs, ϕ_i satisfy homogeneous essential b.cs)

c_i – (unknown) coefficients

Weighting function represented using similar basis

$$w = \sum_{i=1}^{n} \phi_i b_i = \boldsymbol{\phi}\, \boldsymbol{b}$$

Substitute into integral equation which must be satisfied for any b_i to obtain system of n algebraic equations in n unknowns c_i, easily solved.

Finite element method

Problem to be solved

Solve boundary value problem

$$u''(x) + 6x^2 = 0 \quad x \in (0,1)\ , \quad \text{b.cs:}\ \ u(0) = 1\ , \quad u'(1) = -\frac{1}{2}$$

using Galerkin formulation of FEM and 2 elements with linear interpolation.

Analytical solution

$$u''(x) = -6x^2$$

$$u'(x) = -2x^3 + C$$

$$u(x) = -\frac{1}{2}x^4 + Cx + D$$

$$u^{analit} = -\frac{1}{2}x^4 + \frac{3}{2}x + 1$$

Weighted residual method

Global model via WRM

$$R = u''(x) + 6x^2\ , \qquad \int_0^1 w(x)R(x)\mathrm{d}x = 0 \quad \forall w \neq 0$$

$$\int_0^1 wu''\mathrm{d}x + \int_0^1 w\,6x^2\mathrm{d}x = 0 \quad \forall w$$

Note that exact solution is assumed to be C^1 continuous

Weak formulation

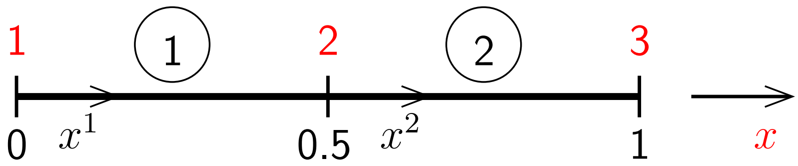
$$-\int_0^1 w'u'\mathrm{d}x + [wu']_0^1 + \int_0^1 w\,6x^2\mathrm{d}x = 0 \quad \forall w \qquad | \cdot (-1)$$

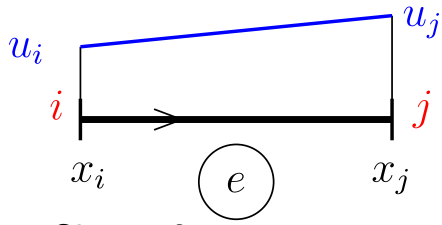
$$\int_0^1 w'u'\mathrm{d}x - w(1)u'(1) + w(0)u'(0) - \int_0^1 w\,6x^2\mathrm{d}x = 0\ , \quad u(0) = 1$$

Note that $u'(1) = -\frac{1}{2}$ and $u'(0)$ is unknown

FE discretization

2 elements with linear interpolation





Topology $e = 1 \quad i = 1 \quad j = 2$

$e = 2 \quad i = 2 \quad j = 3$

Transformation $x^e \in (0, l^e)$

$x = x^e + a^e$

$a^1 = 0, \;\; a^2 = 0.5$

Shape functions

$N_i = 1 - \frac{x^e}{l^e} = 1 - 2x^e$

$N_j = \frac{x^e}{l^e} = 2x^e$

$\boldsymbol{N} = [N_i, N_j]$

$\boldsymbol{d}^e = \begin{bmatrix} u_i \\ u_j \end{bmatrix}$

Bubnov-Galerkin approximation

$$u \approx u^e = \boldsymbol{N}\boldsymbol{d}^e \;, \quad w \approx w^e = \boldsymbol{N}\boldsymbol{b} = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{N}^{\mathrm{T}}$$

Finite element equations

Integral equation for FE

$$\int_0^{l^e} w'u'\mathrm{d}x^e - w(l^e)u'(l^e) + w(0^e)u'(0^e) - \int_0^{l^e} w\; 6x^2\mathrm{d}x^e = 0 \quad \forall w$$

$$\int_0^{l^e} w'u'\mathrm{d}x^e - w(l^e)u'(l^e) + w(0^e)u'(0^e) - \int_0^{l^e} w\; 6(x^e + a^e)^2\mathrm{d}x^e = 0$$

Substitute interpolation $u = \boldsymbol{N}\boldsymbol{d}^e, \quad w = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{N}^{\mathrm{T}}, \quad$ invoke $\forall \boldsymbol{b}$

$$\int_0^{l^e} \boldsymbol{b}^{\mathrm{T}}\boldsymbol{N}'^{\mathrm{T}}\boldsymbol{N}'\boldsymbol{d}^e\mathrm{d}x^e - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{N}^{\mathrm{T}}(l^e)u'(l^e) + \boldsymbol{b}^{\mathrm{T}}\boldsymbol{N}^{\mathrm{T}}(0^e)u'(0^e) - \int_0^{l^e} \boldsymbol{b}^{\mathrm{T}}\boldsymbol{N}^{\mathrm{T}}6(x^e + a^e)^2\mathrm{d}x^e = 0 \quad \forall$$

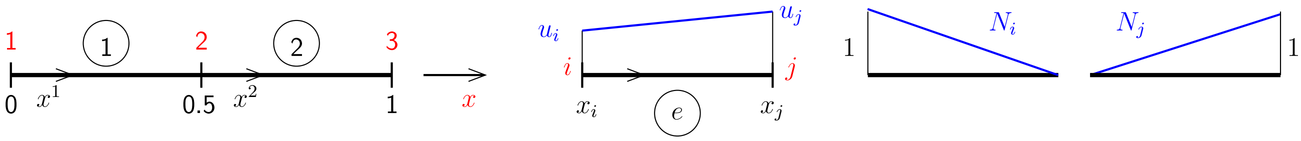
Note that $u'(0^e)$ and $u'(l^e)$ are not approximated

$$\int_0^{l^e} \boldsymbol{N}'^{\mathrm{T}}\boldsymbol{N}'\boldsymbol{d}^e\mathrm{d}x^e - \boldsymbol{N}^{\mathrm{T}}(l^e)u'(l^e) + \boldsymbol{N}^{\mathrm{T}}(0^e)u'(0^e) - \int_0^{l^e} \boldsymbol{N}^{\mathrm{T}}6(x^e + a^e)^2\mathrm{d}x^e = 0$$

Substitute $\boldsymbol{N}^{\mathrm{T}}(l^e) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \;, \quad \boldsymbol{N}^{\mathrm{T}}(0^e) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\int_0^{l^e} \boldsymbol{N}'^{\mathrm{T}}\boldsymbol{N}'\mathrm{d}x^e \; \boldsymbol{d}^e - \begin{bmatrix} -u'(0^e) \\ u'(l^e) \end{bmatrix} - \int_0^{l^e} \boldsymbol{N}^{\mathrm{T}}6(x^e + a^e)^2\mathrm{d}x^e = 0$$

Finite element equations



Finite element matrices

$$\int_0^{l^e} \boldsymbol{N}'^{\mathrm{T}} \boldsymbol{N}' \mathrm{d}x^e \, \boldsymbol{d}^e - \begin{bmatrix} -u'(0^e) \\ u'(l^e) \end{bmatrix} - \int_0^{l^e} \boldsymbol{N}^{\mathrm{T}} 6(x^e + a^e)^2 \mathrm{d}x^e = 0$$

$$\boldsymbol{K}^e = \int_0^{l^e} \boldsymbol{N}'^{\mathrm{T}} \boldsymbol{N}' \mathrm{d}x^e \,, \quad \boldsymbol{p}^e = \int_0^{l^e} \boldsymbol{N}^{\mathrm{T}} 6(x^e + a^e)^2 \mathrm{d}x^e \,, \quad \boldsymbol{p}_b^e = \begin{bmatrix} -u'(0^e) \\ u'(l^e) \end{bmatrix}$$

Note that $\boldsymbol{N}' = [-2\,,\ 2]$

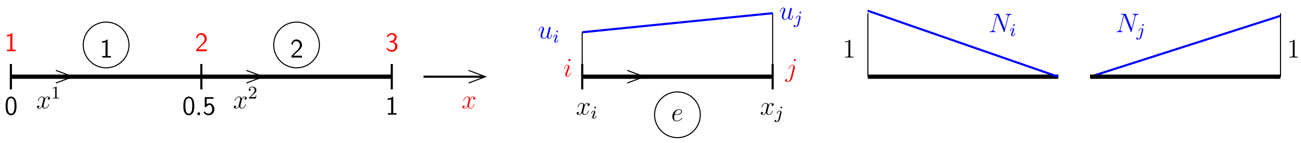
Matrix equation for FE

$$\boldsymbol{K}^e \boldsymbol{d}^e - \boldsymbol{p}_b^e - \boldsymbol{p}^e = 0$$

$$\boldsymbol{K}^e \boldsymbol{d}^e = \boldsymbol{p}^e + \boldsymbol{p}_b^e$$

Numerical model at element level

Computations



Compute matrices for each element

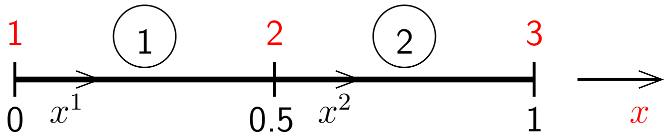
$$\boldsymbol{K}^1 = \boldsymbol{K}^2 = \int_0^{0.5} \begin{bmatrix} -2 \\ 2 \end{bmatrix} \begin{bmatrix} -2 & 2 \end{bmatrix} \mathrm{d}x^e = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$\boldsymbol{p}^1 = \int_0^{0.5} \begin{bmatrix} 1 - 2x^1 \\ 2x^1 \end{bmatrix} 6(x^1)^2 \mathrm{d}x^1 = \begin{bmatrix} 0.0625 \\ 0.1875 \end{bmatrix}$$

$$\boldsymbol{p}^2 = \int_0^{0.5} \begin{bmatrix} 1 - 2x^{(2)} \\ 2x^{(2)} \end{bmatrix} 6(x^{(2)} + 0.5)^2 \mathrm{d}x^{(2)} = \begin{bmatrix} 0.6875 \\ 1.0625 \end{bmatrix}$$

$$\boldsymbol{p}_b^1 = \begin{bmatrix} -u'(0^1) \\ u'(l^1) \end{bmatrix} \,, \quad \boldsymbol{p}_b^2 = \begin{bmatrix} -u'(0^2) \\ u'(l^2) \end{bmatrix}$$

Global set of equations



Assembly

Add element matrices to zeroed global arrays according to topology

$$\boldsymbol{K} = \sum_e \boldsymbol{K}^e\,, \quad \boldsymbol{d} = \sum_e \boldsymbol{d}^e\,, \quad \boldsymbol{p} = \sum_e \boldsymbol{p}^e\,, \quad \boldsymbol{p}_b = \sum_e \boldsymbol{p}_b^e\,,$$

$$\boldsymbol{K}\boldsymbol{d} = \boldsymbol{p} + \boldsymbol{p}_b$$

$$\boldsymbol{K} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2+2 & -2 \\ 0 & -2 & 2 \end{bmatrix} \quad \boldsymbol{d} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \boldsymbol{p} = \begin{bmatrix} 0.0625 \\ 0.8750 \\ 1.0625 \end{bmatrix}$$

$$\boldsymbol{p}_b = \begin{bmatrix} -u'(0^1) \\ u'(l^1) - u'(0^2) \\ u'(l^2) \end{bmatrix} = \begin{bmatrix} -u'(0) \\ 0 \\ u'(1) \end{bmatrix}$$

Boundary conditions and solution

Set of 3 equations in 5 unknowns

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.0625 \\ 0.8750 \\ 1.0625 \end{bmatrix} + \begin{bmatrix} -u'(0) \\ 0 \\ u'(1) \end{bmatrix}$$

but we have boundary conditions $u_1 = u(0) = 1$ and $u'(1) = -0.5$! Notice that until now the solution is independent of the boundary conditions.

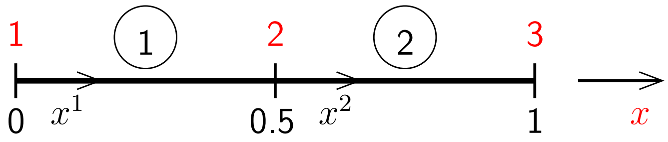
Set of 3 equations in 3 unknowns

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1.0 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.0625 \\ 0.8750 \\ 1.0625 \end{bmatrix} + \begin{bmatrix} -u'(0) \\ 0 \\ -0.5 \end{bmatrix}$$

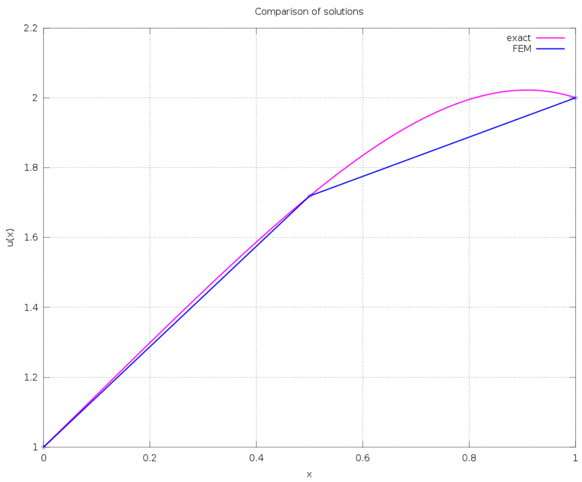
First solve equations 2 and 3, then equation 1 to obtain

$$u_2 = 1.71875\,, \quad u_3 = 2\,, \quad u'(0) = 1.5$$

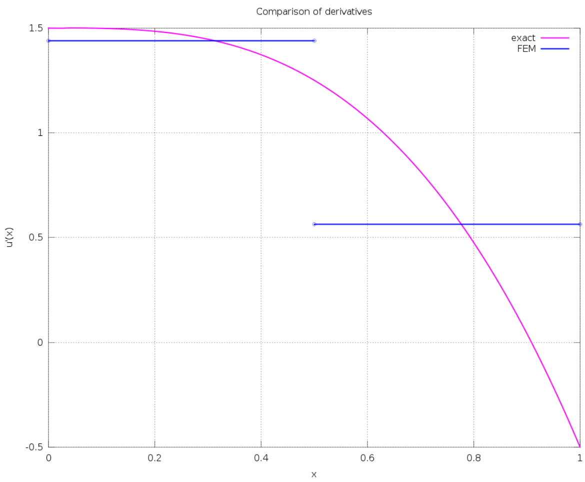
Solution



Comparison of approximate and analytical solutions



Solutions



Solution derivatives