FEM for bar structures (statics)

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Theory - equilibrium equations

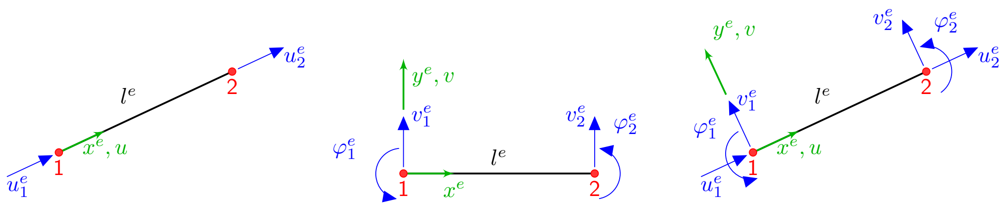
Virtual work principle

$$\delta W_{int} = \delta W_{ext} \quad \forall \delta \mathbf{u}$$

Disassembly into finite elements

$$\delta W_{int} = \sum_e \delta W^e_{int}\,, \quad \delta W_{ext} = \sum_e \delta W^e_{ext}$$

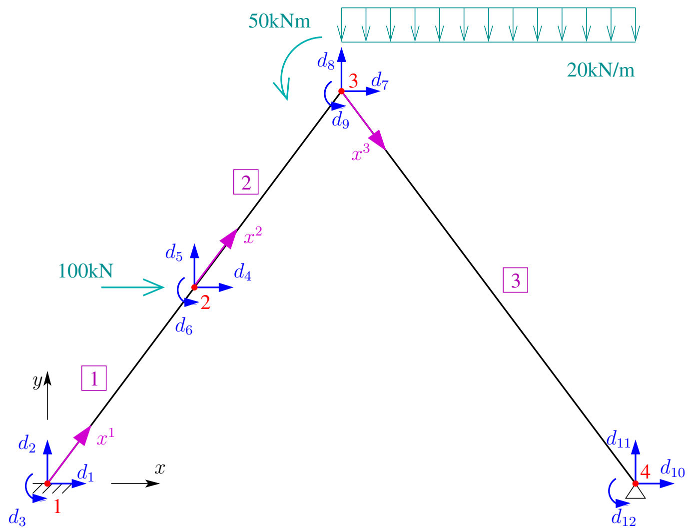
Equivalent to requirement of minimum of total potential energy functional in the space of admissible displacements



truss FE   beam FE   frame FE

Element relations written in local (element) coordinate system

Discretization



Topology (element connectivities)

$$\text{TOP} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \quad \begin{matrix} e=1 \\ e=2 \\ e=3 \end{matrix}$$

Data:

Nodal coordinates

Cross-section stiffnesses EA and EI

Boundary conditions

Loading

Discretization:

NN=4, NE=3, NDOFN=3, NDOF=NN*NDOFN=12

Vector of degees of freedom (dofs):

$$\mathbf{d}^e = \begin{bmatrix} d_1^e \\ d_2^e \\ d_3^e \\ d_4^e \\ d_5^e \\ d_6^e \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \\ d_9 \\ d_{10} \\ d_{11} \\ d_{12} \end{bmatrix}$$

Boundary conditions:

$$d_1 = d_2 = d_3 = d_{10} = d_{11} = 0$$

Theory - equilibrium equations

Variables for frame structures

$\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}$ - displacement vector (fundamental unknown)

$\mathbf{e} = \begin{bmatrix} \epsilon_0 \\ \kappa \end{bmatrix}$ - generalized strain vector ($\mathbf{e} = \mathbf{Lu}$)

$\mathbf{s} = \begin{bmatrix} N \\ M \end{bmatrix}$ - generalized stress vector ($\mathbf{s} = \mathbf{De}$)

$\mathbf{p} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$ - distributed load intensity vector

Galerkin approximation within element

$\mathbf{u}(x^e) = \mathbf{N}(x^e)\mathbf{d}^e\,, \quad \delta\mathbf{u} = \mathbf{N}(x^e)\delta\mathbf{d}^e$

$\mathbf{d}^e$ - degrees of freedom (dofs), i.e. nodal displacements

$\mathbf{e}(x^e) = \mathbf{B}(x^e)\mathbf{d}^e\,, \quad \mathbf{B} = \mathbf{LN}\,, \quad \delta\mathbf{e} = \mathbf{B}(x^e)\delta\mathbf{d}^e$

$\mathbf{s}(x^e) = \mathbf{D}\,\mathbf{e}(x^e) = \mathbf{DB}(x^e)\mathbf{d}^e$

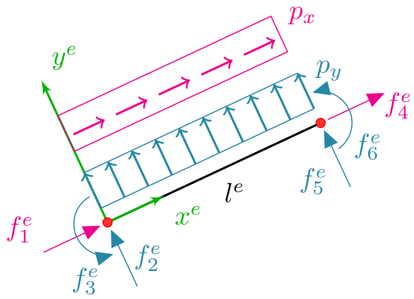
Theory - equilibrium equations

Virtual work principle

$$\delta W^e_{int} = \int_0^{l^e} \delta \mathbf{e}^{\mathrm{T}} \mathbf{s} \, \mathrm{d}x^e$$

$$\delta W^e_{ext} = \int_0^{l^e} \delta \mathbf{u}^{\mathrm{T}} \mathbf{p} \, \mathrm{d}x^e + \delta \mathbf{d}^{e\mathrm{T}} \mathbf{f}^e$$

$\mathbf{f}^e$ - nodal force vector (forces acting on the considered element, which come from elements connected to it at nodes)



$$\mathbf{f}^e = \begin{bmatrix} f_1^e \\ f_2^e \\ f_3^e \\ f_4^e \\ f_5^e \\ f_6^e \end{bmatrix} = \begin{bmatrix} -N_1 \\ Q_1 \\ -M_1 \\ N_2 \\ -Q_2 \\ M_2 \end{bmatrix}$$

Theory - equilibrium equations

Substitute approximation

$$\delta W^e_{int} = \delta \mathbf{d}^{e\mathrm{T}} \int_0^{l^e} \mathbf{B}^{\mathrm{T}} \mathbf{D} \mathbf{B} \, \mathrm{d}x^e \, \mathbf{d}^e = \delta \mathbf{d}^{e\mathrm{T}} \mathbf{K}^e \mathbf{d}^e$$

$\mathbf{K}^e$ - element stiffness matrix

$$\delta W^e_{ext} = \delta \mathbf{d}^{e\mathrm{T}} \int_0^{l^e} \mathbf{N}^{\mathrm{T}} \mathbf{p} \, \mathrm{d}x^e + \delta \mathbf{d}^{e\mathrm{T}} \mathbf{f}^e = \delta \mathbf{d}^{e\mathrm{T}} (\mathbf{z}^e + \mathbf{f}^e)$$

$\mathbf{z}^e$ - equivalent joint loads (substitute nodal forces)

Invoke $\delta W^e_{int} = \delta W^e_{ext} \quad \forall \delta \mathbf{d}^e$

Element balance equations

$$\mathbf{K}^e \mathbf{d}^e = \mathbf{z}^e + \mathbf{f}^e$$

Beam element description

Bending representation

Definitions of displacement, generalized strain and generalized stress

$$\mathbf{u}(x) = [v(x)], \quad \mathbf{e}(x) = [\kappa(x)], \quad \mathbf{s}(x) = [M(x)]$$

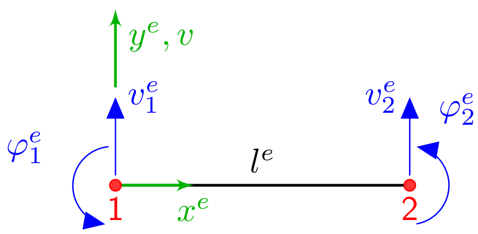
Kinematic and constitutive relations at point
$P(x, y, z) = P(x, 0, 0) = P(x)$ on beam axis

$$\kappa(x) = -\frac{\mathrm{d}^2 v(x)}{\mathrm{d}x^2} \quad \rightarrow \quad \mathbf{e} = \mathbf{Lu}, \quad \mathbf{L} = \left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \right]$$

$$M(x) = EI(x)\ \kappa(x) \quad \rightarrow \quad \mathbf{s} = \mathbf{De}, \quad \mathbf{D} = \left[EI(x) \right]$$

Beam element description

Approximation of deflection

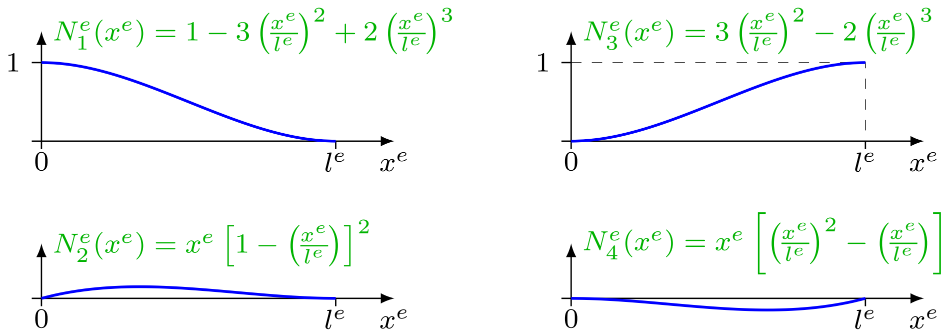


$$NDOF_n = 2, \; NDOF^e = 4$$

$$\underset{[2\times1]}{\mathbf{d}_w} = \{v_w, \varphi_w\}$$

$$\underset{[4\times1]}{\mathbf{d}^e} = \{v_1^e, \varphi_1^e, v_2^e, \varphi_2^e\}$$

$$\underset{[1\times1]}{\mathbf{u}(x^e)} = [v(x^e)] = \underset{[1\times4]}{\mathbf{N}(x^e)} \ \underset{[4\times1]}{\mathbf{d}^e}, \quad \mathbf{N} = [N_1^e \;\; N_2^e \;\; N_3^e \;\; N_4^e]$$



Beam element description

Approximation of curvature and bending moment, stiffness matrix

$$\underset{[1\times 1]}{\mathbf{e}(x^e)} = [\kappa(x^e)] = \mathbf{LN}(x^e)\cdot\mathbf{d}^e = \underset{[1\times 4]}{\mathbf{B}(x^e)}\cdot\underset{[4\times 1]}{\mathbf{d}^e}$$

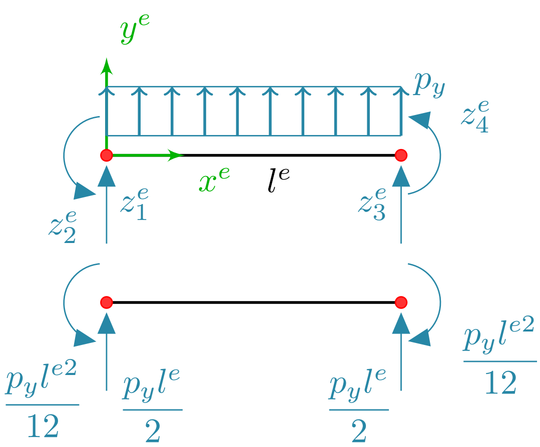
$$\underset{[1\times 1]}{\mathbf{s}(x^e)} = [M(x^e)] = \underset{[1\times 1]}{\mathbf{D}}\cdot\underset{[1\times 4]}{\mathbf{B}(x^e)}\cdot\underset{[4\times 1]}{\mathbf{d}^e}$$

$$\underset{[4\times 4]}{\mathbf{K}^e} = \int_0^{l^e}\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\,\mathrm{d}x^e$$

$$\mathbf{K}^e = \frac{E^e I^e}{l^{e3}}\begin{bmatrix} 12 & 6l^e & -12 & 6l^e \\ 6l^e & 4l^{e2} & -6l^e & 2l^{e2} \\ -12 & -6l^e & 12 & -6l^e \\ 6l^e & 2l^{e2} & -6l^e & 4l^{e2}\end{bmatrix}$$

Beam element description

Computation of substitute nodal forces for constant distributed loading



$$\mathbf{z}^e = \int_0^{l^e}\mathbf{N}^{\mathrm{T}}\left[\; p_y \;\right]\mathrm{d}x^e$$

$$\mathbf{z}^e = \begin{bmatrix} \frac{p_y l^e}{2} \\ \frac{p_y l^{e2}}{12} \\ \frac{p_y l^e}{2} \\ -\frac{p_y l^{e2}}{12}\end{bmatrix}$$

Global balance

Transformation $\mathbf{T}^e$: global $\longrightarrow$ local

Matrices referred to local element axes will be marked by overbar

$$\bar{\mathbf{d}}^e = \mathbf{T}^e \mathbf{d}^e\,, \quad \mathbf{z}^e = \mathbf{T}^{e\mathrm{T}} \bar{\mathbf{z}}^e\,, \quad \mathbf{K}^e = \mathbf{T}^{e\mathrm{T}} \bar{\mathbf{K}}^e \mathbf{T}^e$$

Element balance equations in local coordinates

$$\bar{\mathbf{f}}^e = \bar{\mathbf{K}}^e \bar{\mathbf{d}}^e - \bar{\mathbf{z}}^e$$

Element balance equations in global coordinates

$$\mathbf{f}^e = \mathbf{K}^e \mathbf{d}^e - \mathbf{z}^e$$

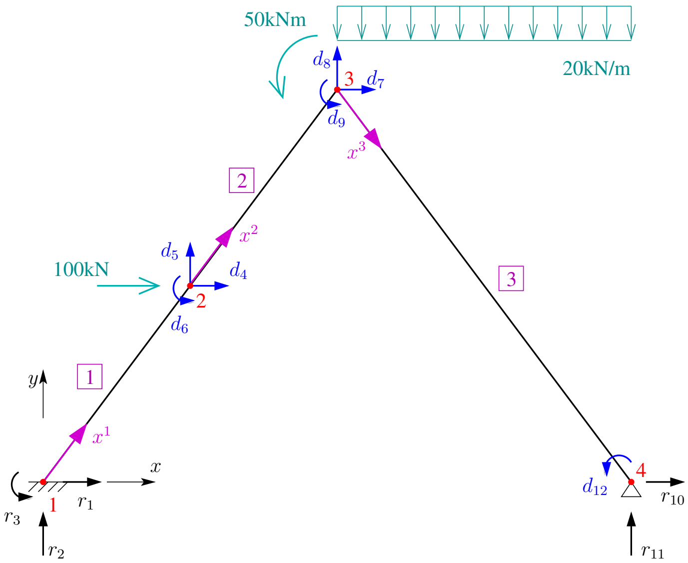
Assembly

$$\mathbf{K} = \sum_e \mathbf{K}^e\,, \quad \mathbf{d} = \sum_e \mathbf{d}^e\,, \quad \mathbf{z} = \sum_e \mathbf{z}^e\,, \quad \mathbf{f} = \sum_e \mathbf{f}^e$$

$$\mathbf{f} = \mathbf{K}\,\mathbf{d} - \mathbf{z} = \mathbf{w} + \mathbf{r}$$

$\mathbf{w}$ - external point load vector
$\mathbf{r}$ - support reaction vector

Computation algorithm



Global vectors:

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \\ d_9 \\ d_{10} \\ d_{11} \\ d_{12} \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \\ r_9 \\ r_{10} \\ r_{11} \\ r_{12} \end{bmatrix}$$

Boundary conditions:

$$d_1 = d_2 = d_3 = d_{10} = d_{11} = 0$$

Hence:

$$r_4 = r_5 = r_6 = r_7 = r_8 = r_9 = r_{12} = 0$$

System equilibrium

$$\mathbf{K}\,\mathbf{d} = \mathbf{w} + \mathbf{z} + \mathbf{r}$$

Element equilibrium

$$\bar{\mathbf{f}}^e = \bar{\mathbf{K}}^e \bar{\mathbf{d}}^e - \bar{\mathbf{z}}^e$$

Algorithm of FE computations for a bar structure

Equilibrium of discretized system (of nodes)

$$\mathbf{K}\,\mathbf{d} = \mathbf{w} + \mathbf{z} + \mathbf{r}$$

plus essential boundary conditions

Statics

1. Discretize (set numbers, axes, topology), prepare input data
2. Compute element matrices $\bar{\mathbf{K}}^e$, $\mathbf{K}^e$, assemble global matrix $\mathbf{K}$
3. Compute element vectors $\bar{\mathbf{z}}^e$, $\mathbf{z}^e$, assemble global vector $\mathbf{z}$, set up point load vector $\mathbf{w}$
4. Solve equation set $\mathbf{K}\mathbf{d} = \mathbf{w} + \mathbf{z} + \mathbf{r}$ taking into account kinematic boundary conditions, i.e. compute unknown nodal displacements in $\mathbf{d}$ and reactions in $\mathbf{r}$

Algorithm of FE computations for a bar structure

Statics (cont'd)

Divide matrices into blocks

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1 + \mathbf{z}_1 \\ \mathbf{w}_2 + \mathbf{z}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}$$

Known displacements $\mathbf{d}_2 = \hat{\mathbf{d}}$, hence $\mathbf{r}_1 = \mathbf{0}$

$\mathbf{K}_{11}\mathbf{d}_1 = \mathbf{w}_1 + \mathbf{z}_1 - \mathbf{K}_{12}\hat{\mathbf{d}} \;\rightarrow\; \mathbf{d}_1 \;\rightarrow\; \mathbf{d}$

$\mathbf{r} = \mathbf{K}\mathbf{d} - \mathbf{z} - \mathbf{w}$

5. Compute nodal forces in elements

   $\mathbf{d} \;\rightarrow\; \mathbf{d}^e \;\rightarrow\; \bar{\mathbf{d}}^e \;\rightarrow\; \bar{\mathbf{f}}^e = \bar{\mathbf{K}}^e\bar{\mathbf{d}}^e - \bar{\mathbf{z}}^e$

   or $\mathbf{d} \;\rightarrow\; \mathbf{d}^e \;\rightarrow\; \mathbf{f}^e = \mathbf{K}^e\mathbf{d}^e - \mathbf{z}^e \;\rightarrow\; \bar{\mathbf{f}}^e$
6. Plot diagrams of section forces, check equilibrium