STATIC FEM ALGORITHM FOR A TRUSS

Piotr Pluciński
e-mail: Piotr.Plucinski@pk.edu.pl

Jerzy Pamin
e-mail: Jerzy.Pamin@pk.edu.pl]

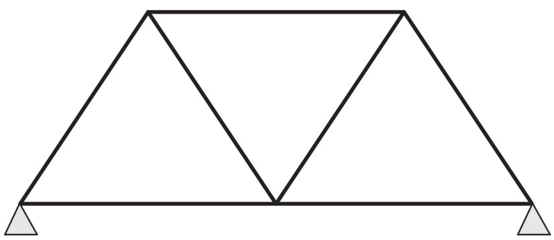
Chair for Computational Engineering
Faculty of Civil Engineering, Cracow University of Technology
URL: www.CCE.pk.edu.pl

Computational model - discretization

Requirements to be satisfied by computational model

In computational model we must guarantee:

- continuity of displacements (at nodes, where elements are connected)
- satisfaction of kinematic constraints
- satisfaction of equilibrium equations for the structure and any substructure (e.g. a node or element)

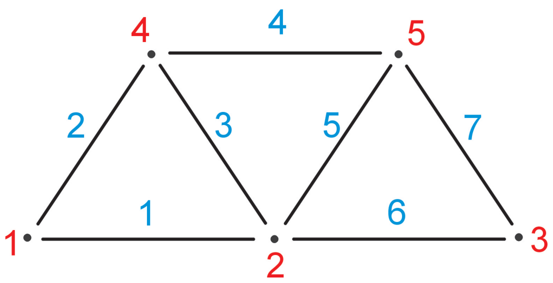


Computational model - discretization

Discretization process - mesh generation

Bar structure is idealized as a discrete system of elements and nodes

- Numbering nodes
- Numbering elements
- Specification of relations between elements and nodes (topology of discrete model)



e	i	j
1	1	2
2	1	4
3	2	4
4	4	5
5	2	5
6	2	3
7	3	5

Truss element description (2D)

Definitions of mechanical variables

Definitions of displacement, strain and cross-section force in the bar under tension/compression

$$\mathbf{u}(x) = \{u(x)\}, \quad \mathbf{e}(x) = \{\varepsilon_0(x)\}, \quad \mathbf{s}(x) = \{N(x)\}$$

Kinematic and physical equation for a point
$P(x,y,z) = P(x,0,0) = P(x)$ on bar axis

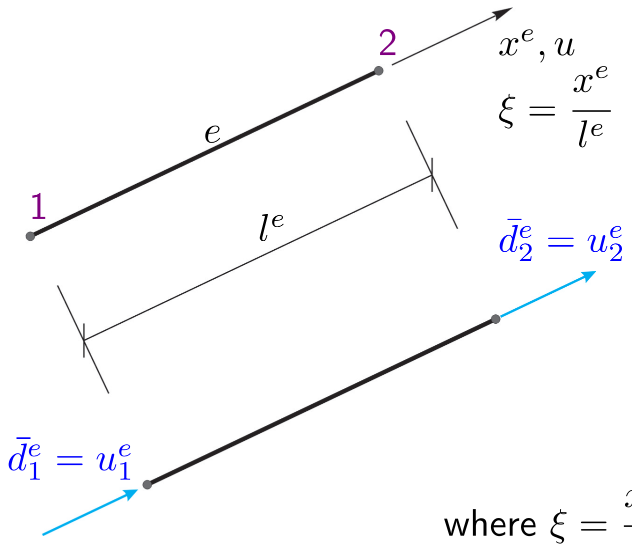
$$\varepsilon_0 = \frac{\mathrm{d}u}{\mathrm{d}x} \quad \rightarrow \quad \mathbf{e} = \mathbf{L}\mathbf{u}, \quad \mathbf{L} = \left[\frac{\mathrm{d}}{\mathrm{d}x}\right]$$

$$N = EA \cdot \varepsilon_0 \quad \rightarrow \quad \mathbf{s} = \mathbf{D}\mathbf{e}, \quad \mathbf{D} = [EA]$$

Truss element description (2D)

Displacement approximation in element (local) coordinate system $0x^e$

Number of local dofs of a node $ndof_w = 1$ and element $ndof^e = 2$.



$$\underset{[1\times 1]}{\mathbf{d}_w} = \{u_w\}$$

$$\underset{[2\times 1]}{\bar{\mathbf{d}}^e} = \{\bar{d}^e_1, \bar{d}^e_2\} = \{u^e_1, u^e_2\}$$

$$\underset{[1\times 1]}{\mathbf{u}(\xi)} = \{u(\xi)\} = \underset{[1\times 2]}{\mathbf{N}(\xi)} \cdot \underset{[2\times 1]}{\bar{\mathbf{d}}^e} =$$

$$= \left[\begin{array}{cc} (1-\xi) & \xi \end{array} \right] \cdot \left[\begin{array}{c} \bar{d}^e_1 \\ \bar{d}^e_2 \end{array} \right]$$

where $\xi = \frac{x^e}{l^e}$ normalized coordinate

Truss element description (2D)

Strain, normal force and stiffness matrix in element (local) coordinate system $0x^e$

$$\underset{[1\times 1]}{\mathbf{e}(\xi)} = \{\varepsilon_0(\xi)\} = \mathbf{L}\mathbf{N}^e(\xi) \cdot \bar{\mathbf{d}}^e = \underset{[1\times 2]}{\mathbf{B}(\xi)} \cdot \underset{[2\times 1]}{\bar{\mathbf{d}}^e} = \left[\begin{array}{cc} -\frac{1}{l^e} & \frac{1}{l^e} \end{array} \right] \cdot \left[\begin{array}{c} \bar{d}^e_1 \\ \bar{d}^e_2 \end{array} \right]$$

$$\underset{[1\times 1]}{\mathbf{s}(\xi)} = \{N(\xi)\} = \underset{[1\times 1]}{\mathbf{D}} \cdot \underset{[1\times 2]}{\mathbf{B}(\xi)} \cdot \underset{[2\times 1]}{\bar{\mathbf{d}}^e} = [EA]^e \cdot \left[\begin{array}{cc} -\frac{1}{l^e} & \frac{1}{l^e} \end{array} \right] \cdot \left[\begin{array}{c} \bar{d}^e_1 \\ \bar{d}^e_2 \end{array} \right]$$

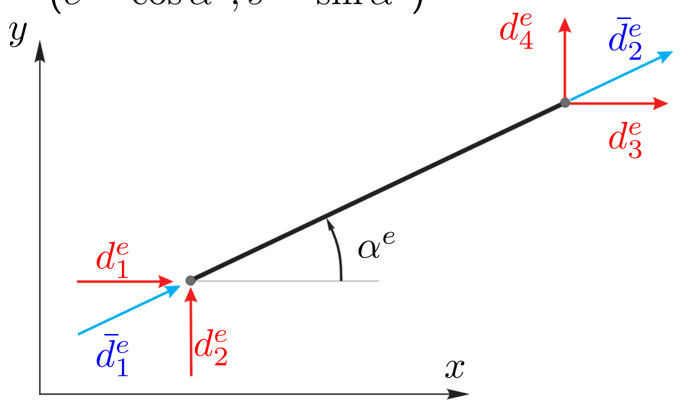
Element stiffness matrix in local coordinate system

$$\underset{[2\times 2]}{\bar{\mathbf{K}}^e} = \int_0^{l^e} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\,\mathrm{d}x = \left(\frac{EA}{L}\right)^e \cdot \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right]$$

Truss element description (2D)

Description of truss element in global coordinate set $0XY$

Number of global dofs of a node $NDOF_w = 2$ and element $NDOF^e = 4$
$(c = \cos\alpha^e, s = \sin\alpha^e)$



$$\underset{[2\times1]}{\mathbf{d}_w} = \{u_w, v_w\}$$

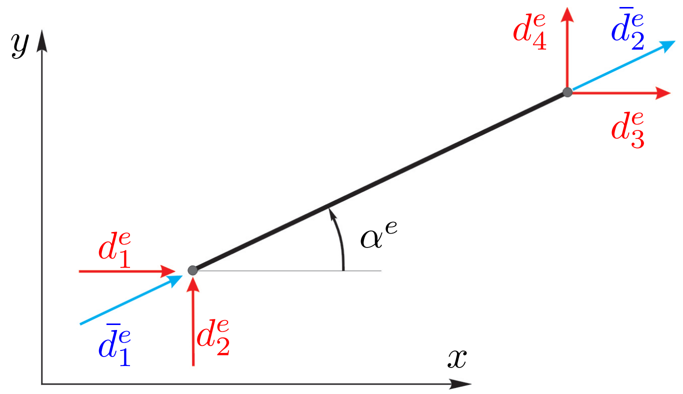
$$\underset{[4\times1]}{\mathbf{d}^e} = \{d_1, d_2, d_3, d_4\} = \{u_1, v_1, u_2, v_2\}$$

$$\underset{[2\times1]}{\bar{\mathbf{d}}^e} = \begin{bmatrix} \bar{d}_1 \\ \bar{d}_2 \end{bmatrix}^e = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}^e \cdot \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}^e = \underset{[2\times4]}{\mathbf{T}^e} \cdot \underset{[4\times1]}{\mathbf{d}^e}$$

$$\underset{[4\times1]}{\mathbf{d}^e} = \underset{[4\times2]}{\mathbf{T}^{e\mathrm{T}}} \cdot \underset{[2\times1]}{\bar{\mathbf{d}}^e}$$

Truss element description (2D)

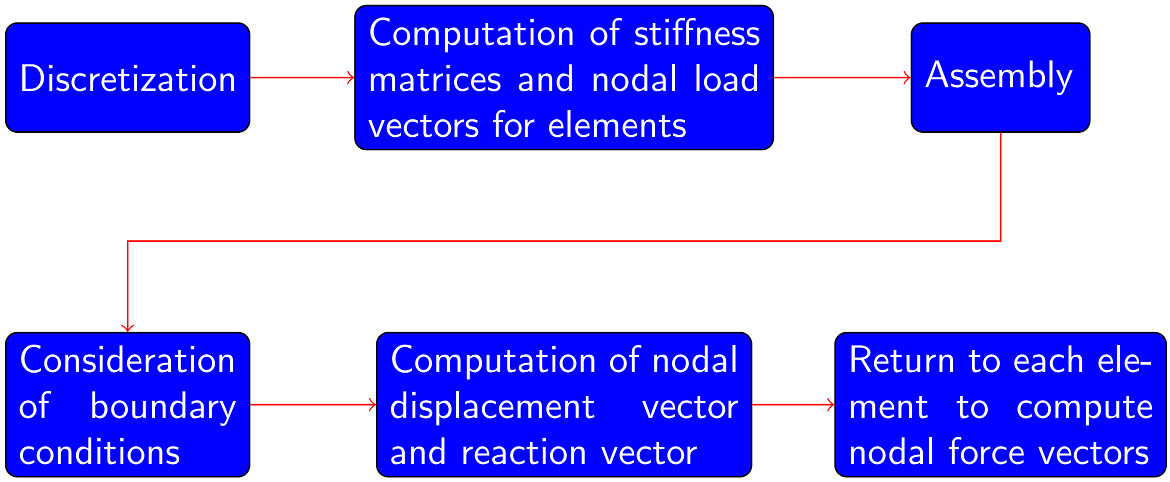
Transformation of element stiffness matrix



Global stiffness matrix: $(c = \cos\alpha^e, s = \sin\alpha^e)$

$$\underset{[4\times4]}{\mathbf{K}^e} = (\mathbf{T}^\mathrm{T}\bar{\mathbf{K}}\mathbf{T})^e = \left(\frac{EA}{L}\right)^e \cdot \begin{bmatrix} cc & cs & -cc & -cs \\ cs & ss & -cs & -ss \\ -cc & -cs & cc & cs \\ -cs & -ss & cs & ss \end{bmatrix}^e$$

Flowchart of FEM algorithm for statics



System balance

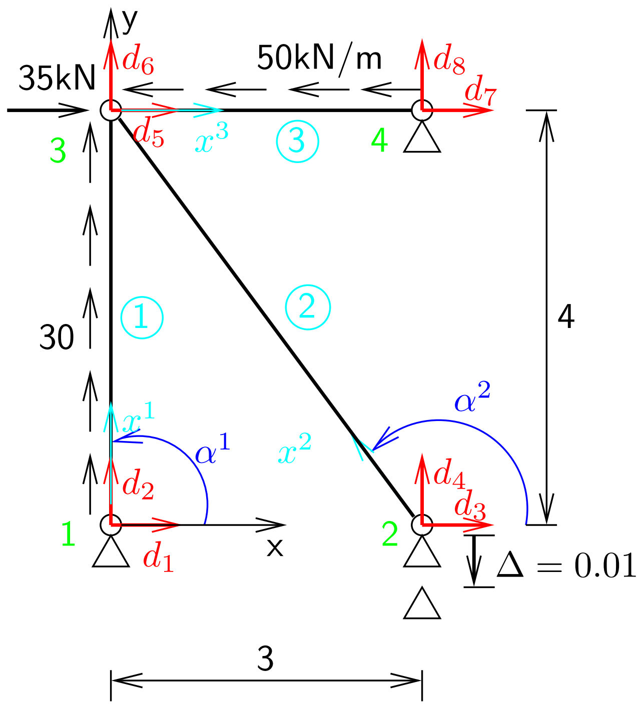
$$\mathbf{K}\,\mathbf{d} = \mathbf{w} + \mathbf{z} + \mathbf{r} \quad \text{(oraz w.b.)}$$

Element balance

$$\bar{\mathbf{f}}^e = \bar{\mathbf{K}}^e \bar{\mathbf{d}}^e - \bar{\mathbf{z}}^e$$

Example computations for 2D truss

Problem definition and discretization



Longit. stiffness $EA = 10^4$kN

Elem.1: $l = 4,\ c = 0,\ s = 1$

Elem.2: $l = 5,\ -c = 0.6,\ s = 0.8$

Elem.3: $l = 3,\ c = 1,\ s = 0$

Incidence matrix

$$\text{TOP} = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$$

$$\mathbf{T}^e = \begin{bmatrix} \cos\alpha^e & \sin\alpha^e & 0 & 0 \\ 0 & 0 & \cos\alpha^e & \sin\alpha^e \end{bmatrix}$$

$$\underset{[4\times4]}{\mathbf{K}^e} = \frac{EA}{L^e} \begin{bmatrix} cc & cs & -cc & -cs \\ cs & ss & -cs & -ss \\ -cc & -cs & cc & cs \\ -cs & -ss & cs & ss \end{bmatrix}$$

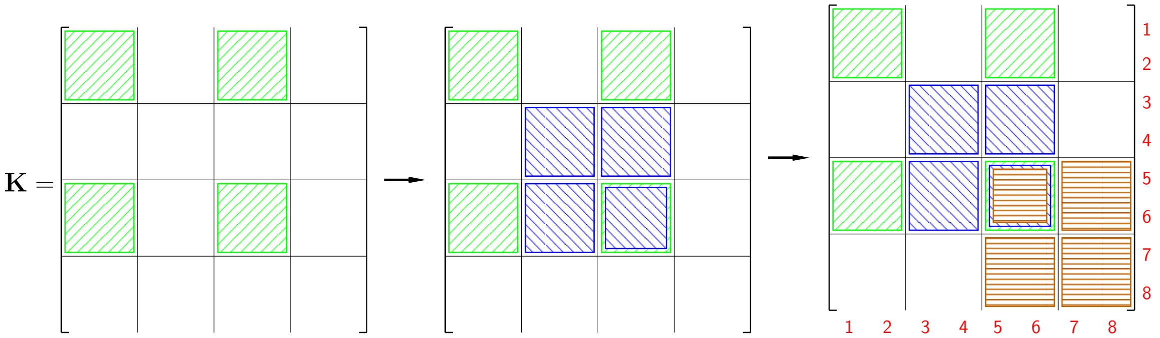
Example computations for 2D truss

Stiffness matrices in global coordinate set for all elements and assembly

$$\mathbf{K}^1 = \frac{10^4}{4}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 2500 & 0 & -2500 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -2500 & 0 & 2500 \end{array}\right] \begin{matrix} 1 \\ 2 \\ 5 \\ 6 \end{matrix}$$

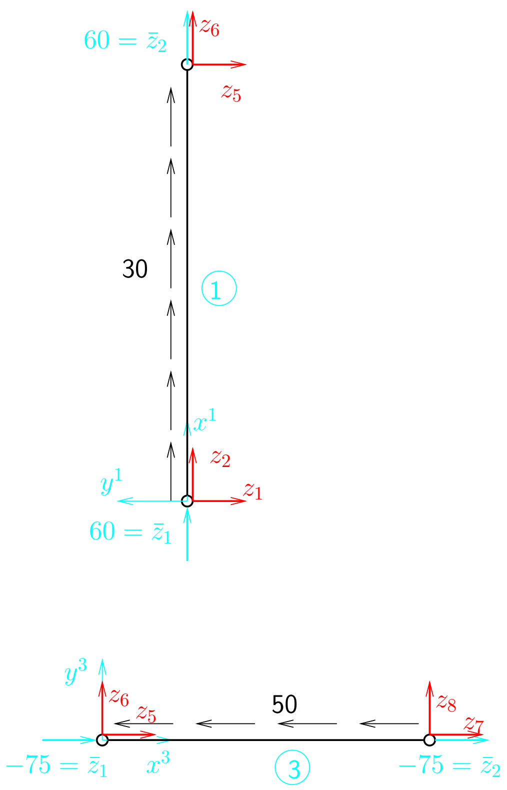
$$\mathbf{K}^2 = \frac{10^4}{5}\begin{bmatrix} 0.36 & -0.48 & -0.36 & 0.48 \\ -0.48 & 0.64 & 0.48 & -0.64 \\ -0.36 & 0.48 & 0.36 & -0.48 \\ 0.48 & -0.64 & -0.48 & 0.64 \end{bmatrix} = \begin{bmatrix} 720 & -960 & -720 & 960 \\ -960 & 1280 & 960 & -1280 \\ -720 & 960 & 720 & -960 \\ 960 & -1280 & -960 & 1280 \end{bmatrix} \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

$$\mathbf{K}^3 = \frac{10^4}{3}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3333 & 0 & -3333 & 0 \\ 0 & 0 & 0 & 0 \\ -3333 & 0 & 3333 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 5 \\ 6 \\ 7 \\ 8 \end{matrix}$$



Example computations for 2D truss

Substitute nodal force vectors for all loaded elements and assembly



$$\bar{\mathbf{z}}^e = \int_0^{l^e} \mathbf{N}^T p_x(x) \mathrm{d}x \, , \mathbf{N} = \left[1 - \frac{x^e}{l^e}, \frac{x^e}{l^e}\right]$$

$$\bar{\mathbf{z}}^1 = \int_0^4 \begin{bmatrix} 1 - \frac{x}{4} \\ \frac{x}{4} \end{bmatrix} \cdot 30 \, \mathrm{d}x = \begin{bmatrix} 60 \\ 60 \end{bmatrix}$$

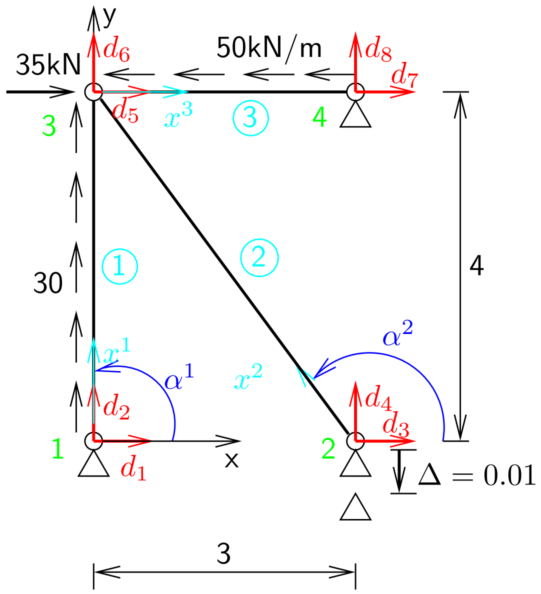
$$\mathbf{z}^1 = (\mathbf{T}^1)^T \bar{\mathbf{z}}^1 = \begin{bmatrix} 0 \\ 60 \\ 0 \\ 60 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 5 \\ 6 \end{matrix}$$

$$\mathbf{z}^2 = 0 \, , \quad \mathbf{z}^3 = \begin{bmatrix} -75 \\ 0 \\ -75 \\ 0 \end{bmatrix} \begin{matrix} 5 \\ 6 \\ 7 \\ 8 \end{matrix}$$

$$\mathbf{z} = \sum_e \mathbf{z}^e = \begin{bmatrix} 0 \\ 60 \\ 0 \\ 0 \\ -75 \\ 60 \\ -75 \\ 0 \end{bmatrix}$$

Example computations for 2D truss

Set of equations, boundary conditions, solution



$$\mathbf{K}\mathbf{d} = \mathbf{z} + \mathbf{w} + \mathbf{r}$$

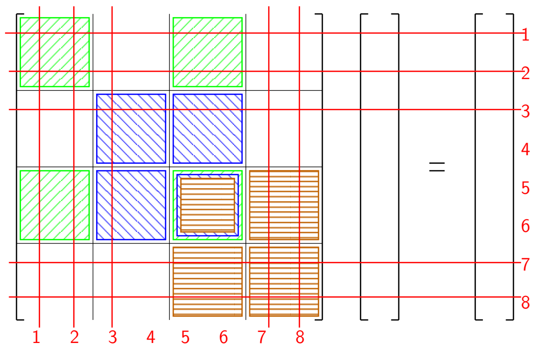
$$\mathbf{w}^T = \{0, 0, 0, 0, 35, 0, 0, 0\}$$

$$\mathbf{r}^T = \{R_1, R_2, R_3, R_4, 0, 0, R_7, R_8\}$$

Boundary conditions:

$$d_1 = d_2 = d_3 = d_7 = d_8 = 0\,,\ d_4 = -0.01$$

Cross out rows and columns for which $d_i = 0$ to obtain 3×3 set



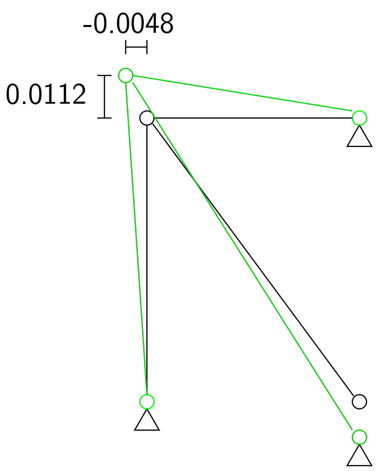
$$\hat{\mathbf{K}}\hat{\mathbf{d}} = \hat{\mathbf{z}} + \hat{\mathbf{w}} + \hat{\mathbf{r}}$$

Example computations for 2D truss

Solution, nodal displacements

$$\begin{bmatrix} 1280 & 960 & -1280 \\ 960 & 0+720+3333 & 0-960+0 \\ -1280 & 0-960+0 & 2500+1280+0 \end{bmatrix} \begin{bmatrix} -0.01 \\ d_5 \\ d_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -75 \\ 60 \end{bmatrix} + \begin{bmatrix} 0 \\ 35 \\ 0 \end{bmatrix} + \begin{bmatrix} R_4 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} 4 \\ 5 \\ 6 \end{matrix}$$

$$\begin{bmatrix} 1280 & 960 & -1280 \\ 960 & 4053 & -960 \\ -1280 & -960 & 3780 \end{bmatrix} \begin{bmatrix} -0.01 \\ d_5 \\ d_6 \end{bmatrix} = \begin{bmatrix} R_4 \\ -40 \\ 60 \end{bmatrix}$$



$$\mathbf{d}_5 = -0.0048\,,\ \mathbf{d}_6 = 0.0112$$

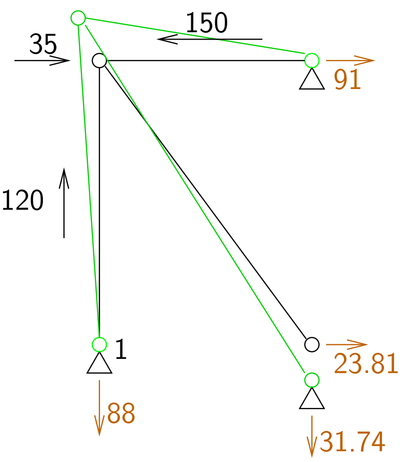
$$\mathbf{d} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.01 \\ -0.0048 \\ 0.0112 \\ 0 \\ 0 \end{bmatrix}$$

Example computations for 2D truss

Solution, support reactions (notice truncation errors)

$$\mathbf{r} = \mathbf{K}\,\mathbf{d} - \mathbf{z} - \mathbf{w}$$

$$\mathbf{r} = \mathbf{K}\begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.01 \\ -0.0048 \\ 0.0112 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 60 \\ 0 \\ 0 \\ -40 \\ 60 \\ -75 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -88.00 \\ 23.81 \\ -31.74 \\ \cancel{0.19}\ 0 \\ \cancel{-0.26}\ 0 \\ 91.00 \\ 0 \end{bmatrix}$$



Check equilibrium:

$$\sum X = -150+35+91.00+23.81 = -0.19$$

$$\sum Y = 120 - 88.00 - 31.74 = 0.26$$

$$\sum M_1 = -150{\cdot}4+35{\cdot}4+91.00{\cdot}4+31.74{\cdot}3 = -0.78$$

Example computations for 2D truss

Return to elements to compute nodal forces

Element 1

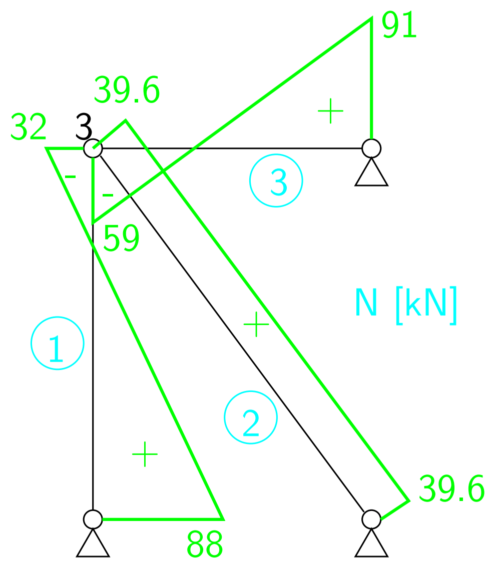
$$\mathbf{d}^1 = \begin{bmatrix} d_1 \\ d_2 \\ d_5 \\ d_6 \end{bmatrix}, \quad \bar{\mathbf{f}}^1 = \mathbf{T}^1(\mathbf{K}^1\mathbf{d}^1 - \mathbf{z}^1) \quad \bar{\mathbf{f}}^1 = \begin{bmatrix} -88.00 \\ -32.00 \end{bmatrix}$$

Element 2

$$\mathbf{d}^2 = \begin{bmatrix} d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix}, \quad \bar{\mathbf{f}}^2 = \mathbf{T}^2(\mathbf{K}^2\mathbf{d}^2) \quad \bar{\mathbf{f}}^2 = \begin{bmatrix} -39.60 \\ 39.60 \end{bmatrix}$$

Element 3

$$\mathbf{d}^3 = \begin{bmatrix} d_5 \\ d_6 \\ d_7 \\ d_8 \end{bmatrix}, \quad \bar{\mathbf{f}}^3 = \mathbf{T}^3(\mathbf{K}^3\mathbf{d}^3 - \mathbf{z}^3) \quad \bar{\mathbf{f}}^3 = \begin{bmatrix} 59.00 \\ 91.00 \end{bmatrix}$$

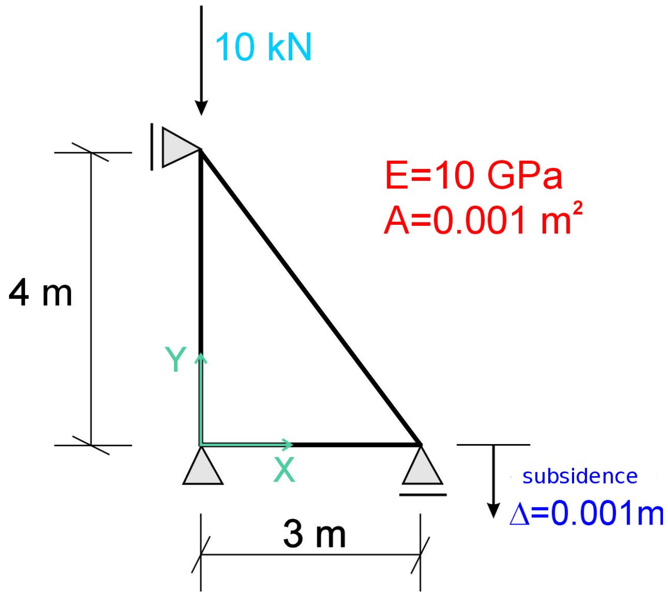


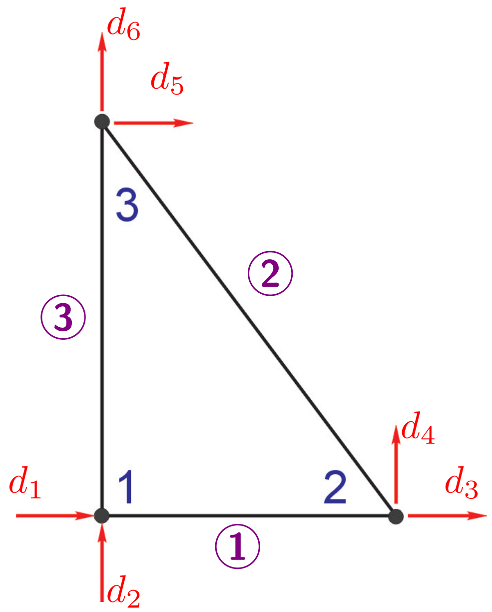
Normal force diagram

Check equilibrium of node 3

2D truss - second example

Problem definition and discretization





2D truss - second example

Input data

Stiffness matrix $\bar{\mathbf{K}}^e$

$$\bar{\mathbf{K}}^e(EA, l^e) = \begin{bmatrix} \frac{EA}{l^e} & -\frac{EA}{l^e} \\ -\frac{EA}{l^e} & \frac{EA}{l^e} \end{bmatrix}$$

Incidence matrix

$$\text{TOP} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 3 \end{bmatrix}$$

Longitudinal stiffness

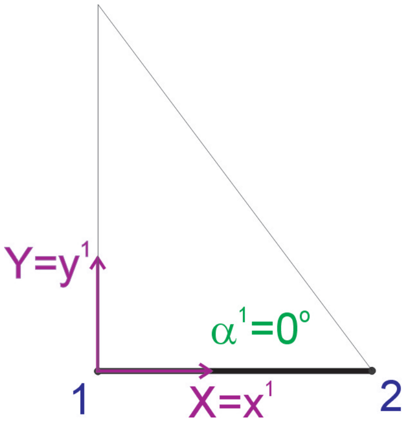
$$EA = 1 \cdot 10^4$$

Transformation matrix $\mathbf{T}^e$

$$\mathbf{T}^e(\cos\alpha, \sin\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \end{bmatrix}$$

2D truss - second example

Computation of transformation matrix and stiffness matrix in local and global coordinate set for element 1



$$x^{(1)} = 3, \quad y^{(1)} = 0, \quad l^{(1)} = \sqrt{{x^{(1)}}^2 + {y^{(1)}}^2}$$

$$\cos\alpha^1 = \frac{x^{(1)}}{l^{(1)}}, \quad \sin\alpha^1 = \frac{y^{(1)}}{l^{(1)}}$$

$$\mathbf{T}^1 = \mathbf{T}^e(\cos\alpha^1, \sin\alpha^1)$$

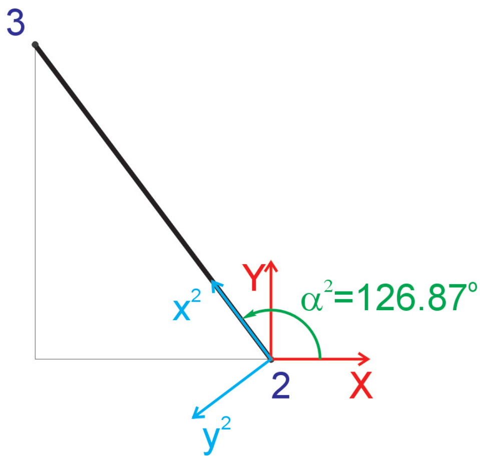
$$\mathbf{T}^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\bar{\mathbf{K}}^1 = \bar{\mathbf{K}}^e(EA, l^{(1)}) \qquad \bar{\mathbf{K}}^1 = \begin{bmatrix} 3.3333 & -3.3333 \\ -3.3333 & 3.3333 \end{bmatrix} \cdot 10^3$$

$$\mathbf{K}^1 = {\mathbf{T}^1}^{\mathrm{T}} \bar{\mathbf{K}}^1 \mathbf{T}^1 \qquad \mathbf{K}^1 = \begin{bmatrix} 3.3333 & 0 & -3.3333 & 0 \\ 0 & 0 & 0 & 0 \\ -3.3333 & 0 & 3.3333 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot 10^3$$

2D truss - second example

Computation of transformation matrix and stiffness matrix in local and global coordinate set for element 2



$$x^{(2)} = -3, \quad y^{(2)} = 4, \quad l^{(2)} = \sqrt{{x^{(2)}}^2 + {y^{(2)}}^2}$$

$$\cos\alpha^2 = \frac{x^{(2)}}{l^{(2)}}, \quad \sin\alpha^2 = \frac{y^{(2)}}{l^{(2)}}$$

$$\mathbf{T}^2 = \mathbf{T}^e(\cos\alpha^2, \sin\alpha^2)$$

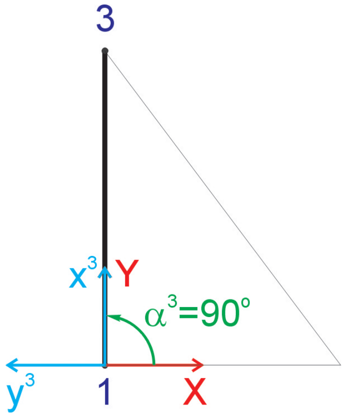
$$\mathbf{T}^2 = \begin{bmatrix} -0.6 & 0.8 & 0 & 0 \\ 0 & 0 & -0.6 & 0.8 \end{bmatrix}$$

$$\bar{\mathbf{K}}^2 = \bar{\mathbf{K}}^e(EA, l^{(2)}) \qquad \bar{\mathbf{K}}^2 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \cdot 10^3$$

$$\mathbf{K}^2 = {\mathbf{T}^2}^{\mathrm{T}} \bar{\mathbf{K}}^2 \mathbf{T}^2 \qquad \mathbf{K}^2 = \begin{bmatrix} 0.72 & -0.96 & -0.72 & 0.96 \\ -0.96 & 1.28 & 0.96 & -1.28 \\ -0.72 & 0.96 & 0.72 & -0.96 \\ 0.96 & -1.28 & -0.96 & 1.28 \end{bmatrix} \cdot 10^3$$

2D truss - second example

Computation of transformation matrix and stiffness matrix in local and global coordinate set for element 3



$$x^{(3)} = 0, \quad y^{(3)} = 4, \quad l^{(3)} = \sqrt{{x^{(3)}}^2 + {y^{(3)}}^2}$$

$$\cos\alpha^3 = \frac{x^{(3)}}{l^{(3)}}, \quad \sin\alpha^3 = \frac{y^{(3)}}{l^{(3)}}$$

$$\mathbf{T}^3 = \mathbf{T}^e(\cos\alpha^3, \sin\alpha^3)$$

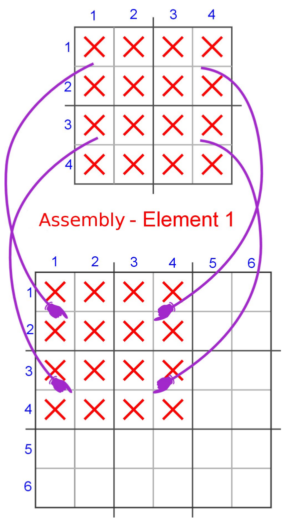
$$\mathbf{T}^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\bar{\mathbf{K}}^3 = \bar{\mathbf{K}}^e(EA, l^{(3)}) \qquad \bar{\mathbf{K}}^3 = \begin{bmatrix} 2.5 & -2.5 \\ -2.5 & 2.5 \end{bmatrix} \cdot 10^3$$

$$\mathbf{K}^3 = {\mathbf{T}^3}^{\mathrm{T}} \bar{\mathbf{K}}^3 \mathbf{T}^3 \qquad \mathbf{K}^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2.5 & 0 & -2.5 \\ 0 & 0 & 0 & 0 \\ 0 & -2.5 & 0 & 2.5 \end{bmatrix} \cdot 10^3$$

2D truss - second example

Assembly of stiffness matrix for element 1 into global stiffness matrix

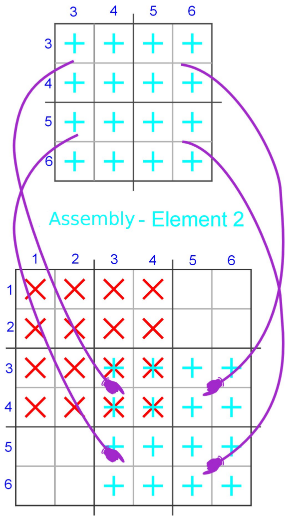


$$\mathbf{K} = \begin{bmatrix} K^1_{11} & K^1_{12} & K^1_{13} & K^1_{14} & 0 & 0 \\ K^1_{21} & K^1_{22} & K^1_{23} & K^1_{24} & 0 & 0 \\ K^1_{31} & K^1_{32} & K^1_{33} & K^1_{34} & 0 & 0 \\ K^1_{41} & K^1_{42} & K^1_{43} & K^1_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 3.3333 & 0 & -3.3333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -3.3333 & 0 & 3.3333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot 10^3$$

2D truss - second example

Assembly of stiffness matrix for element 2 into global stiffness matrix

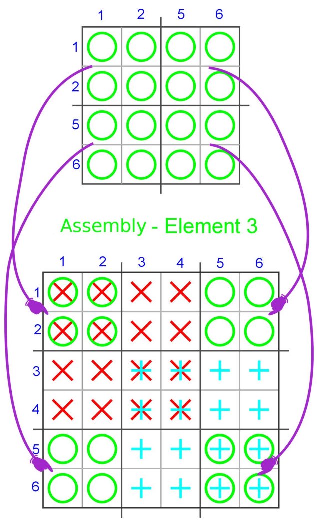


$$\mathbf{K} = \begin{bmatrix} K^1_{11} & K^1_{12} & K^1_{13} & K^1_{14} & 0 & 0 \\ K^1_{21} & K^1_{22} & K^1_{23} & K^1_{24} & 0 & 0 \\ K^1_{31} & K^1_{32} & K^1_{33}+K^2_{11} & K^1_{34}+K^2_{12} & K^2_{13} & K^2_{14} \\ K^1_{41} & K^1_{42} & K^1_{43}+K^2_{21} & K^1_{44}+K^2_{22} & K^2_{23} & K^2_{24} \\ 0 & 0 & K^2_{31} & K^2_{32} & K^2_{33} & K^2_{34} \\ 0 & 0 & K^2_{41} & K^2_{42} & K^2_{43} & K^2_{44} \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 3.3333 & 0 & -3.3333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -3.3333 & 0 & 4.0533 & -0.96 & -0.72 & 0.96 \\ 0 & 0 & -0.96 & 1.28 & 0.96 & -1.28 \\ 0 & 0 & -0.72 & 0.96 & 0.72 & -0.96 \\ 0 & 0 & 0.96 & -1.28 & -0.96 & 1.28 \end{bmatrix} \cdot 10^3$$

2D truss - second example

Assembly of stiffness matrix for element 3 into global stiffness matrix

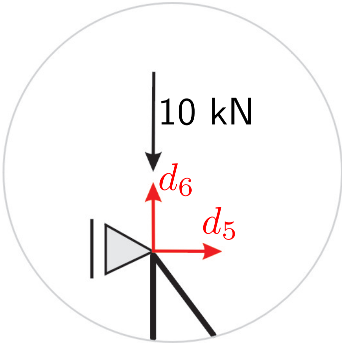


$$\mathbf{K} = \begin{bmatrix} K^1_{11}+K^3_{11} & K^1_{12}+K^3_{12} & K^1_{13} & K^1_{14} & K^3_{13} & K^3_{14} \\ K^1_{21}+K^3_{21} & K^1_{22}+K^3_{22} & K^1_{23} & K^1_{24} & K^3_{23} & K^3_{24} \\ K^1_{31} & K^1_{32} & K^1_{33}+K^2_{11} & K^1_{34}+K^2_{12} & K^2_{13} & K^2_{14} \\ K^1_{41} & K^1_{42} & K^1_{43}+K^2_{21} & K^1_{44}+K^2_{22} & K^2_{23} & K^2_{24} \\ K^3_{31} & K^3_{32} & K^2_{31} & K^2_{32} & K^2_{33}+K^3_{33} & K^2_{34}+K^3_{34} \\ K^3_{41} & K^3_{42} & K^2_{41} & K^2_{42} & K^2_{43}+K^3_{43} & K^2_{44}+K^3_{44} \end{bmatrix}$$

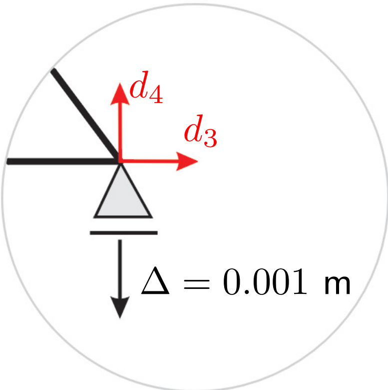
$$\mathbf{K} = \begin{bmatrix} 3.3333 & 0 & -3.3333 & 0 & 0 & 0 \\ 0 & 2.5 & 0 & 0 & 0 & -2.5 \\ -3.3333 & 0 & 4.0533 & -0.96 & -0.72 & 0.96 \\ 0 & 0 & -0.96 & 1.28 & 0.96 & -1.28 \\ 0 & 0 & -0.72 & 0.96 & 0.72 & -0.96 \\ 0 & -2.5 & 0.96 & -1.28 & -0.96 & 3.78 \end{bmatrix} \cdot 10^3$$

2D truss - second example

Load vector and substitute load vector due to imposed displacement Δ



$$\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -10 \end{bmatrix}$$

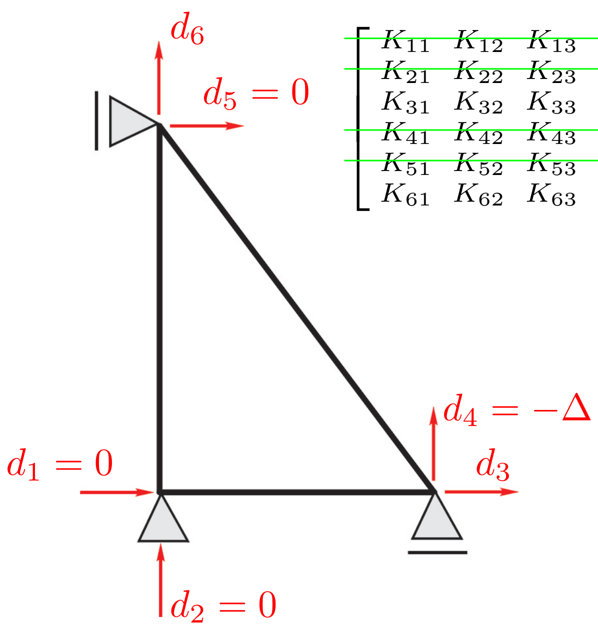


$$\mathbf{d}_{\mathsf{bc}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.001 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{f} = \mathbf{w} - \mathbf{K}\ \mathbf{d}_{\mathsf{bc}}$$

2D truss - second example

Imposition of boundary conditions $\mathbf{K} \rightarrow \hat{\mathbf{K}}, \mathbf{f} \rightarrow \hat{\mathbf{f}}$



$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ d_3 \\ -\Delta \\ 0 \\ d_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -10 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \\ 0 \\ R_4 \\ R_5 \\ 0 \end{bmatrix}$$

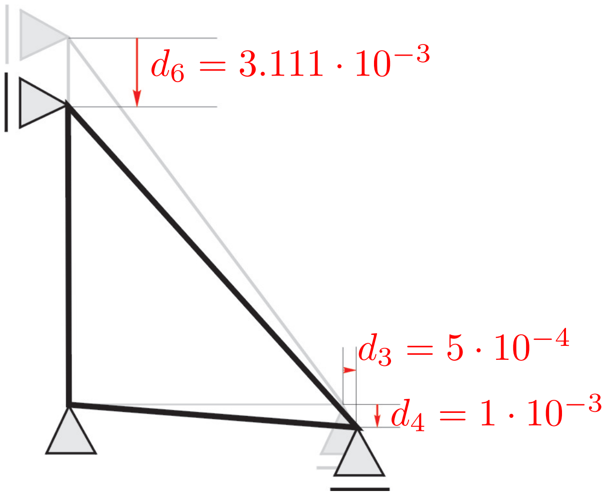
$$\hat{\mathbf{K}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{33} & 0 & 0 & K_{36} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & K_{63} & 0 & 0 & K_{66} \end{bmatrix}, \quad \hat{\mathbf{f}} = \begin{bmatrix} 0 \\ 0 \\ f_3 \\ 0 \\ 0 \\ f_6 \end{bmatrix}$$

2D truss - second example

Computation of nodal displacements

$$\mathbf{d} = \hat{\mathbf{K}}^{-1}\,\hat{\mathbf{f}} + \mathbf{d}_{\mathsf{bc}}$$

$$\mathbf{d} = \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ -1 \\ 0 \\ -3.1111 \end{bmatrix} \cdot 10^{-3}$$

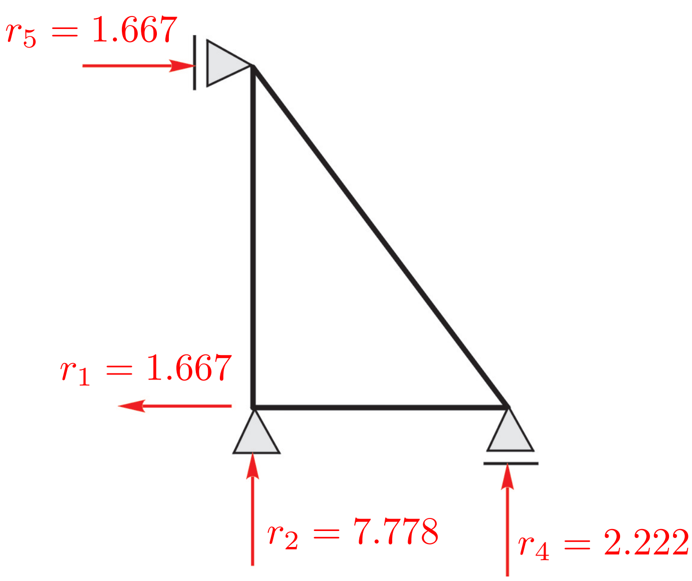


2D truss - second example

Determination of support reactions

$$\mathbf{r} = \mathbf{K}\ \mathbf{d} - \mathbf{w}$$

$$\mathbf{r} = \begin{bmatrix} -1.6667 \\ 7.7778 \\ 0 \\ 2.2222 \\ 1.6667 \\ 0 \end{bmatrix}$$



2D truss - second example

Return to element to compute nodal forces - diagram of nodal forces

Element 1

$$\mathbf{d}^1 = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}, \quad \bar{\mathbf{f}}^1 = \mathbf{T}^1(\mathbf{K}^1\mathbf{d}^1)$$

$$\bar{\mathbf{f}}^1 = \begin{bmatrix} -1.6667 \\ 1.6667 \end{bmatrix}$$

Element 2

$$\mathbf{d}^2 = \begin{bmatrix} d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix}, \quad \bar{\mathbf{f}}^2 = \mathbf{T}^2(\mathbf{K}^2\mathbf{d}^2)$$

$$\bar{\mathbf{f}}^2 = \begin{bmatrix} 2.7778 \\ -2.7778 \end{bmatrix}$$

Element 3

$$\mathbf{d}^3 = \begin{bmatrix} d_1 \\ d_2 \\ d_5 \\ d_6 \end{bmatrix}, \quad \bar{\mathbf{f}}^3 = \mathbf{T}^3(\mathbf{K}^3\mathbf{d}^3)$$

$$\bar{\mathbf{f}}^3 = \begin{bmatrix} 7.7778 \\ -7.7778 \end{bmatrix}$$