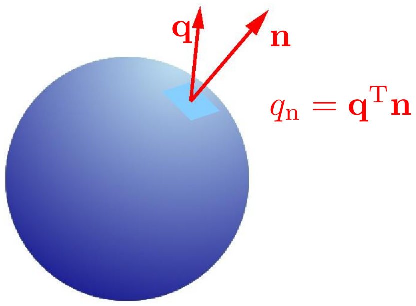
FEM for stationary heat flow

Piotr Pluciński
e-mail: Piotr.Plucinski@pk.edu.pl
Jerzy Pamin
e-mail: Jerzy.Pamin@pk.edu.pl

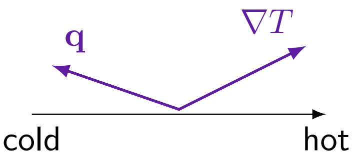
Chair for Computational Engineering
Faculty of Civil Engineering, Cracow University of Technology
URL: www.CCE.pk.edu.pl

Lecture contents

Stationary heat flow in 3D



$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$



Fundamental unknown - temperature T

- Fourier's law of heat conduction
  $\mathbf{q} = -\mathbf{D}\,\nabla T$
- heat flux density vector
  $\mathbf{q} = \{q_x \; q_y \; q_z\}$ [W/m^2]
- temperature gradient
  $\mathrm{grad}T = \nabla T = \left\{ \frac{\partial T}{\partial x} \;\; \frac{\partial T}{\partial y} \;\; \frac{\partial T}{\partial z} \right\}$ [K/m]
- heat conduction matrix
  $\mathbf{D} = \{k_{ij}\}$ [W/(mK)]

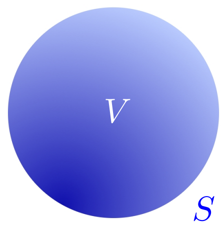
Flux density grows with temperature gradient.
Heat flows from higher to lower temperature.

Heat energy balance

Heat generated equal to heat flowing out

$$\int_V f \mathrm{d}V = \int_S q_\mathrm{n} \mathrm{d}S \qquad \forall\, V$$

f – heat source density – energy supplied to the body per unit volume and time [W/m^3]



Using Green-Gauss-Ostrogradsky theorem about integration by parts

$$\int_S q_\mathrm{n} \mathrm{d}S = \int_S \mathbf{q}^\mathrm{T}\mathbf{n}\,\mathrm{d}S = \int_V \mathrm{div}\mathbf{q}\,\mathrm{d}V = \int_V \left\{ \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right\} \mathrm{d}V$$

$$\int_V f \mathrm{d}V = \int_V \nabla^\mathrm{T}\mathbf{q}\,\mathrm{d}V \;\; \forall\, V \quad \Rightarrow \quad \nabla^\mathrm{T}\mathbf{q} = f \;\; \forall \mathbf{x} \in V$$

Heat flow equations

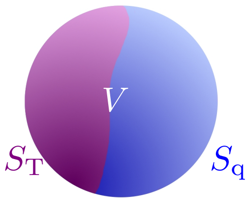
Heat conduction equation (strong formulation)

$$\nabla^{\mathrm{T}}(\mathbf{D}\nabla T) + f = 0 \quad \forall \mathbf{x} \in V$$

+ boundary conditions

$q_{\mathrm{n}} = \mathbf{q}^{\mathrm{T}}\mathbf{n} = \widehat{q}$ on S_{q} – natural b.c. (Neumann)

$T = \widehat{T}$ on S_{T} – essential b.c. (Dirichlet)



Conduction matrix for isotropic materials $\mathbf{D} = k\mathbf{I}$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{f}{k} = 0 \qquad \text{– Poisson equation}$$

For isotropic materials without heat source

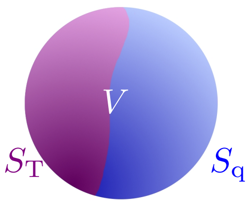
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \qquad \text{– Laplace equation}$$

Stationary heat flow in 3D

Weighted residual method

$$\int_V w \left(\nabla^{\mathrm{T}}(\mathbf{D}\nabla T) + f\right) \mathrm{d}V = 0 \qquad \forall w \neq 0$$

$$\int_V w\nabla^{\mathrm{T}}(\mathbf{D}\nabla T)\mathrm{d}V + \int_V wf\mathrm{d}V = 0$$



Weak formulation

$$-\int_V (\nabla w)^{\mathrm{T}}\mathbf{D}\nabla T\mathrm{d}V + \int_S w\left(\overset{-\mathbf{q}}{\boxed{\mathbf{D}\nabla T}}\right)^{\mathrm{T}}\mathbf{n}\mathrm{d}S + \int_V wf\mathrm{d}V = 0$$

$$-\int_V (\nabla w)^{\mathrm{T}}\mathbf{D}\nabla T\mathrm{d}V - \int_S w\,\overset{q_{\mathrm{n}}}{\boxed{\mathbf{q}^{\mathrm{T}}\mathbf{n}}}\,\mathrm{d}S + \int_V wf\mathrm{d}V = 0$$

$$\int_V (\nabla w)^{\mathrm{T}}\mathbf{D}\nabla T\mathrm{d}V = -\int_{S_{\mathrm{q}}} w\,\underset{\text{natural b.c.}}{\boxed{\widehat{q}}}\,\mathrm{d}S - \int_{S_{\mathrm{T}}} w\,\underset{\text{secondary unknown}}{\boxed{q_{\mathrm{n}}}}\,\mathrm{d}S + \int_V wf\mathrm{d}V \quad \forall w$$

Stationary heat flow in 3D

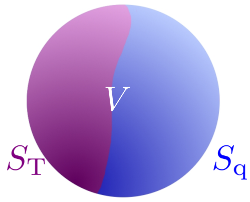
Strong formulation

$$\nabla^{\mathrm{T}}(\mathbf{D}\nabla T) + f = 0 \quad \forall \mathbf{x} \in V$$

+ boundary conditions

$$q_{\mathrm{n}} = \mathbf{q}^{\mathrm{T}}\mathbf{n} = \widehat{q} \qquad \text{on } S_{\mathrm{q}}$$

$$T = \widehat{T} \qquad \text{on } S_{\mathrm{T}}$$



Weak formulation

$$\int_V (\nabla w)^{\mathrm{T}}\mathbf{D}\nabla T \mathrm{d}V = -\int_{S_{\mathrm{q}}} w\widehat{q}\,\mathrm{d}S - \int_{S_{\mathrm{T}}} w q_{\mathrm{n}}\mathrm{d}S + \int_V w f \mathrm{d}V \quad \forall w \neq 0$$

+ boundary condition

$$T = \widehat{T} \quad \text{on } S_{\mathrm{T}}$$

Stationary heat flow in 3D

Set of FE equations

- $T = \mathbf{N}\boldsymbol{\Theta}$ – approximated temperature function
- $\mathbf{N}$ – shape function vector ((global approximation)
- $\boldsymbol{\Theta}$ – nodal temperature (dof) vector
- $\nabla T = \mathbf{B}\boldsymbol{\Theta}$ – approximated temperature gradient function
- $\mathbf{B} = \nabla\mathbf{N}$ – shape function derivative matrix
- $w = \mathbf{w}^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}$ – approximation of weight function ($\nabla w = \mathbf{B}\mathbf{w}$)

$$\int_V (\nabla w)^{\mathrm{T}}\mathbf{D}\nabla T \mathrm{d}V = -\int_{S_{\mathrm{q}}} w\widehat{q}\,\mathrm{d}S - \int_{S_{\mathrm{T}}} w q_{\mathrm{n}}\mathrm{d}S + \int_V w f \mathrm{d}V \quad \forall \mathbf{w}^{\mathrm{T}}$$

$$\mathbf{K}\boldsymbol{\Theta} = \mathbf{f}_{\mathrm{b}} + \mathbf{f}$$

$$\mathbf{K} = \int_V \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\mathrm{d}V, \quad \mathbf{f}_{\mathrm{b}} = -\int_{S_{\mathrm{q}}} \mathbf{N}^{\mathrm{T}}\widehat{q}\,\mathrm{d}S - \int_{S_{\mathrm{T}}} \mathbf{N}^{\mathrm{T}} q_{\mathrm{n}}\,\mathrm{d}S, \quad \mathbf{f} = \int_V \mathbf{N}^{\mathrm{T}} f\,\mathrm{d}V$$

Selection of approximation functions

Fundamental steps in FE procedure

1. Build a strong formulation of a problem
2. Convert the formulation into a weak format
3. Select approximation of unknown function
4. Select weighting function (usually Galerkin approach)

Selection of approximation functions in 1D

1 Strong formulation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(Ak\frac{\mathrm{d}T}{\mathrm{d}x}\right)+f=0$$

+ boundary conditions

$$q_x=\widehat{q} \qquad \text{at } x_{\mathrm{q}} \ (\text{ e.g. } x_{\mathrm{q}}=0)$$

$$T=\widehat{T} \qquad \text{at } x_{\mathrm{T}} \ (\text{ e.g. } x_{\mathrm{T}}=l)$$

2 Conversion into weak formulation

$$\int_0^l \frac{\mathrm{d}w}{\mathrm{d}x}\left(Ak\frac{\mathrm{d}T}{\mathrm{d}x}\right)\mathrm{d}x=(wA)\Big|_{x=0}\widehat{q}-(wAq_x)\Big|_{x=l}+\int_0^l wf\,\mathrm{d}x$$

+ b.c.

$$T=\widehat{T} \qquad \text{at } x_{\mathrm{T}} \ (\text{ e.g. } x_{\mathrm{T}}=l)$$

Selection of approximation functions in 1D

3 Selection of approximation functions

Linear function

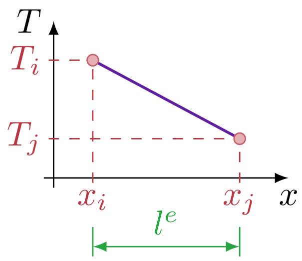
$$T^e(x) = \alpha_1 + \alpha_2 x = \mathbf{\Phi}\boldsymbol{\alpha}^e$$

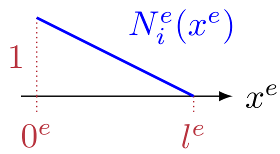
$$\mathbf{\Phi} = [1 \ x], \qquad \boldsymbol{\alpha}^e = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

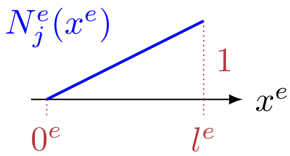
$$T^e(x) = N_i^e(x^e)T_i + N_j^e(x^e)T_j = \mathbf{N}^e\mathbf{\Theta}^e$$

$$\mathbf{N}^e = [N_i^e(x^e) \ N_j^e(x^e)], \qquad \mathbf{\Theta}^e = \begin{bmatrix} T_i \\ T_j \end{bmatrix}$$

$$\frac{\mathrm{d}T^e}{\mathrm{d}x^e} = \mathbf{B}^e\mathbf{\Theta}^e, \qquad \mathbf{B}^e = \frac{\mathrm{d}\mathbf{N}^e}{\mathrm{d}x^e} = \begin{bmatrix} \frac{\mathrm{d}N_i^e}{\mathrm{d}x^e} & \frac{\mathrm{d}N_j^e}{\mathrm{d}x^e} \end{bmatrix}$$







Selection of approximation functions in 1D

3 Selection of approximation functions

Quadratic function

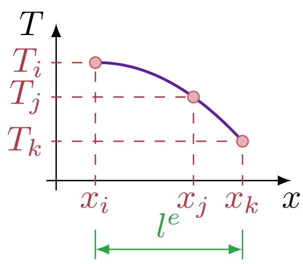
$$T^e(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 = \mathbf{\Phi}\boldsymbol{\alpha}^e$$

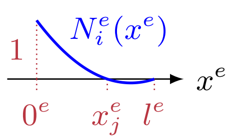
$$\mathbf{\Phi} = [1 \ x \ x^2], \qquad \boldsymbol{\alpha}^e = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

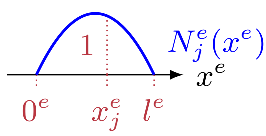
$$T^e(x) = N_i^e(x^e)T_i + N_j^e(x^e)T_j + N_k^e(x^e)T_k$$
$$= \mathbf{N}^e\mathbf{\Theta}^e$$

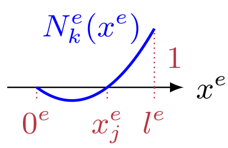
$$\mathbf{N}^e = [N_i^e(x^e) \ N_j^e(x^e) \ N_k^e(x^e)], \quad \mathbf{\Theta}^e = \begin{bmatrix} T_i \\ T_j \\ T_k \end{bmatrix}$$

$$\frac{\mathrm{d}T^e}{\mathrm{d}x^e} = \mathbf{B}^e\mathbf{\Theta}^e, \qquad \mathbf{B}^e = \frac{\mathrm{d}\mathbf{N}^e}{\mathrm{d}x^e} = \begin{bmatrix} \frac{\mathrm{d}N_i^e}{\mathrm{d}x^e} & \frac{\mathrm{d}N_j^e}{\mathrm{d}x^e} & \frac{\mathrm{d}N_k^e}{\mathrm{d}x^e} \end{bmatrix}$$









Selection of approximation functions in 2D

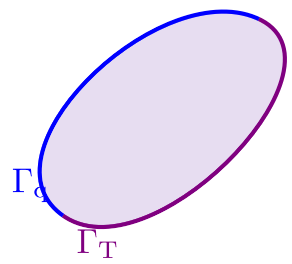
1 Strong formulation

$$\nabla^{\mathrm{T}}(\mathbf{D}\nabla T) + f = 0 \quad \forall \mathbf{x} \in A$$

+ boundary conditions

$$q_{\mathrm{n}} = \mathbf{q}^{\mathrm{T}}\mathbf{n} = \widehat{q} \qquad \text{on } \Gamma_{\mathrm{q}}$$

$$T = \widehat{T} \qquad \text{on } \Gamma_{\mathrm{T}}$$



2 Conversion into weak formulation (h - configuration thickness)

$$\int_A (\nabla w)^{\mathrm{T}}\mathbf{D}h\nabla T \mathrm{d}A = -\int_{\Gamma_{\mathrm{q}}} wh\widehat{q}\mathrm{d}\Gamma - \int_{\Gamma_{\mathrm{T}}} whq_{\mathrm{n}}\mathrm{d}\Gamma + \int_A whf\mathrm{d}A$$

+ boundary condition

$$T = \widehat{T} \quad \text{on } \Gamma_{\mathrm{T}}$$

Selection of approximation functions in 2D

3 Selection of approximation functions

Three-node element

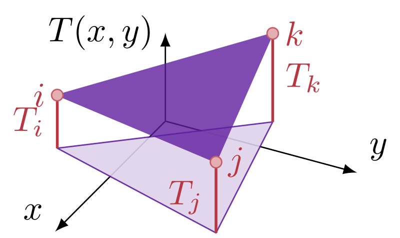
$$T^e(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y = \mathbf{\Phi}\boldsymbol{\alpha}^e$$

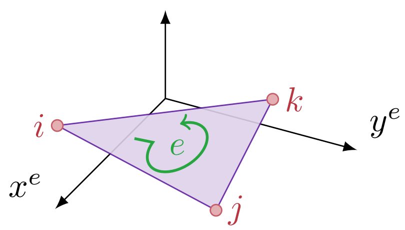
$$\mathbf{\Phi} = [1 \; x \; y], \qquad \boldsymbol{\alpha}^e = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$T^e(x,y) = N_i^e(x^e,y^e)T_i + N_j^e(x^e,y^e)T_j + \\ + N_k^e(x^e,y^e)T_k = \mathbf{N}^e\mathbf{\Theta}^e$$

$$\mathbf{N}^e = [N_i^e(x^e,y^e) \; N_j^e(x^e,y^e) \; N_k^e(x^e,y^e)],$$

$$\mathbf{\Theta}^e = \begin{bmatrix} T_i \\ T_j \\ T_k \end{bmatrix}$$





Selection of approximation functions in 2D

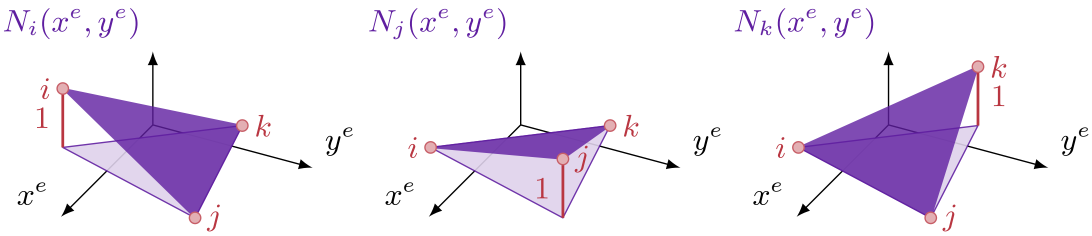
3 Selection of approximation functions

Three-node element

$$\mathbf{N}^e = [N_i^e(x^e,y^e)\ N_j^e(x^e,y^e)\ N_k^e(x^e,y^e)]$$

e.g. for $N_i(x^e,y^e)$

$$N_i(x_i^e,y_i^e)=1$$
$$N_i(x_j^e,y_j^e)=0$$
$$N_i(x_k^e,y_k^e)=0$$



Determination of shape functions $N_i(x^e,y^e)=\alpha_{1i}+\alpha_{2i}x^e+\alpha_{3i}y^e$

$$\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{bmatrix} \alpha_{1i} \\ \alpha_{2i} \\ \alpha_{3i} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \begin{aligned} \alpha_{1i} &= \frac{x_j y_k - x_k y_j}{2P_\triangle} \\ \alpha_{2i} &= \frac{y_j - y_k}{2P_\triangle} \\ \alpha_{3i} &= \frac{x_k - x_j}{2P_\triangle} \end{aligned}$$

Selection of approximation functions in 2D

Pascal triangle – three-node element

$$\begin{array}{ccccccccc} &&&& 1 &&&& \\ &&& x && y &&& \\ && x^2 && xy && y^2 && \\ & x^3 && x^2y && xy^2 && y^3 & \\ x^4 && x^3y && x^2y^2 && xy^3 && y^4 \end{array}$$

Pascal triangle – six-node element

$$\begin{array}{ccccccccc} &&&& 1 &&&& \\ &&& x && y &&& \\ && x^2 && xy && y^2 && \\ & x^3 && x^2y && xy^2 && y^3 & \\ x^4 && x^3y && x^2y^2 && xy^3 && y^4 \end{array}$$

Selection of approximation functions in 2D

Convergence conditions - requirements for FE approximation

- completeness – approximation must be able to represent a constant field and a constant field gradient
- continuity (conforming FE) – approximation must be continuous along interelement edges

Selection of approximation functions in 2D

3 Selection of approximation functions

Four-node element

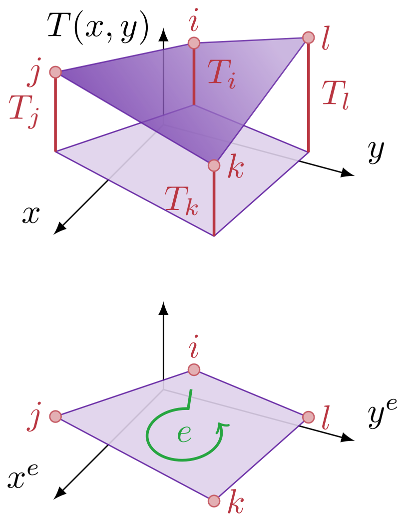
$$T^e(x,y) = \alpha_1^e + \alpha_2^e x + \alpha_3^e y + \alpha_4^e xy = \mathbf{\Phi}\boldsymbol{\alpha}^e$$

$$\mathbf{\Phi} = [1 \; x \; y \; xy], \qquad \boldsymbol{\alpha}^e = \begin{bmatrix} \alpha_1^e \\ \alpha_2^e \\ \alpha_3^e \\ \alpha_4^e \end{bmatrix}$$

$$T^e(x,y) = N_i^e(x^e,y^e)T_i + N_j^e(x^e,y^e)T_j + \\ + N_k^e(x^e,y^e)T_k + N_l^e(x^e,y^e)T_l = \mathbf{N}^e\mathbf{\Theta}^e$$

$$\mathbf{N}^e = [N_i^e(x^e,y^e) \; N_j^e(x^e,y^e) \; N_k^e(x^e,y^e) \; N_l^e(x^e,y^e)]$$

$$\mathbf{\Theta}^e = \{T_i \; T_j \; T_k \; T_l\}$$



Selection of approximation functions in 2D

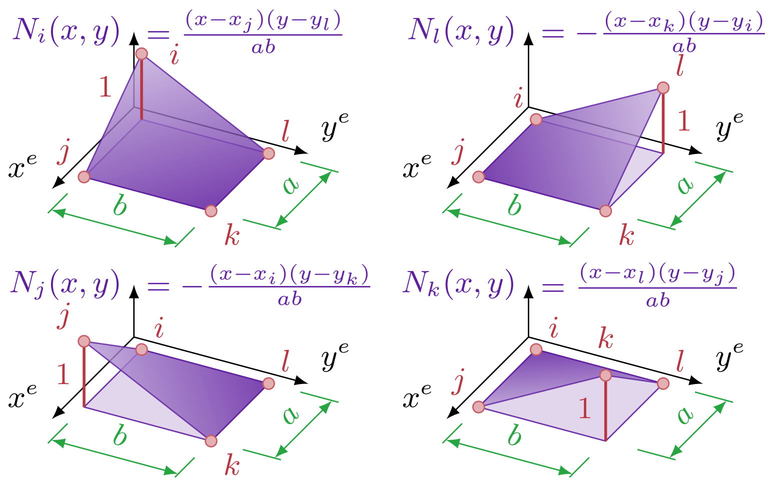
3 Selection of approximation functions

Four-node element (rectangular)

$$\mathbf{N}^e = [N_i^e(x^e,y^e)\ N_j^e(x^e,y^e)\ N_k^e(x^e,y^e)\ N_l^e(x^e,y^e)]$$

e.g. for $N_i(x^e,y^e)$

$$N_i(x_i^e,y_i^e)=1$$
$$N_i(x_j^e,y_j^e)=0$$
$$N_i(x_k^e,y_k^e)=0$$
$$N_i(x_l^e,y_l^e)=0$$



$$\nabla T^e = \mathbf{B}^e\boldsymbol{\Theta}^e$$

$$\mathbf{B}^e = \begin{bmatrix} \frac{\partial \mathbf{N}^e}{\partial x^e} \\ \frac{\partial \mathbf{N}^e}{\partial y^e} \end{bmatrix}$$

Selection of approximation functions in 2D

Pascal triangle – four-node element

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & x & & y & & & \\ & & x^2 & & xy & & y^2 & & \\ & x^3 & & x^2y & & xy^2 & & y^3 & \\ x^4 & & x^3y & & x^2y^2 & & xy^3 & & y^4 \end{array}$$

Pascal triangle – eight-node element

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & x & & y & & & \\ & & x^2 & & xy & & y^2 & & \\ & x^3 & & x^2y & & xy^2 & & y^3 & \\ x^4 & & x^3y & & x^2y^2 & & xy^3 & & y^4 \end{array}$$

Selection of approximation functions in 3D

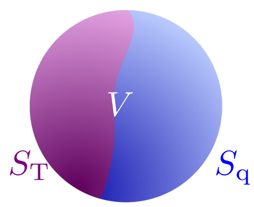
1 Strong formulation

$$\nabla^{\mathrm{T}}(\mathbf{D}\nabla T) + f = 0 \quad \forall \mathbf{x} \in V$$

+ boundary conditions

$$q_{\mathrm{n}} = \mathbf{q}^{\mathrm{T}}\mathbf{n} = \widehat{q} \qquad \text{on } S_{\mathrm{q}}$$

$$T = \widehat{T} \qquad \text{on } S_{\mathrm{T}}$$



2 Conversion to weak formulation

$$\int_V (\nabla w)^{\mathrm{T}} \mathbf{D} \nabla T \mathrm{d}V = -\int_{S_{\mathrm{q}}} w\widehat{q}\mathrm{d}S - \int_{S_{\mathrm{T}}} w q_{\mathrm{n}} \mathrm{d}S + \int_V w f \mathrm{d}V$$

+ boundary condition

$$T = \widehat{T} \quad \text{on } S_{\mathrm{T}}$$

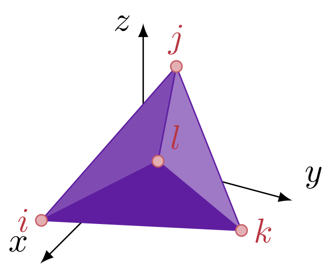
Selection of approximation functions in 3D

3 Selection of approximation functions

Tetrahedral element

$$T^e(x,y,z) = \alpha_1^e + \alpha_2^e x + \alpha_3^e y + \alpha_4^e z$$

$$\begin{aligned} T^e(x,y,z) &= N_i^e(x^e,y^e,z^e)T_i + N_j^e(x^e,y^e,z^e)T_j \\ &+ N_k^e(x^e,y^e,z^e)T_k + N_l^e(x^e,y^e,z^e)T_l \\ &= \mathbf{N}^e \boldsymbol{\Theta}^e \end{aligned}$$



Hexahedral element

$$\begin{aligned} T^e(x,y,z) &= \alpha_1^e + \alpha_2^e x + \alpha_3^e y + \alpha_4^e z + \\ &+ \alpha_5^e xy + \alpha_6^e yz + \alpha_7^e xz + \alpha_8^e xyz \end{aligned}$$