FEM solution for example problem of stationary heat flow

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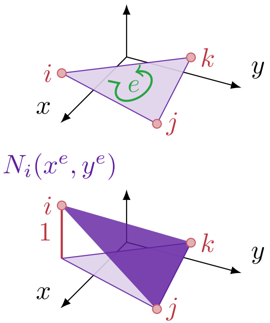
Determination of shape functions for 3-node element



Shape function $N_i(x,y) = \alpha_{1i} + \alpha_{2i}x^e + \alpha_{3i}y^e$

$$\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{bmatrix} \alpha_{1i} \\ \alpha_{2i} \\ \alpha_{3i} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

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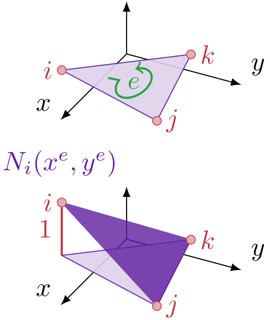

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$$W = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = 2P_\triangle$$

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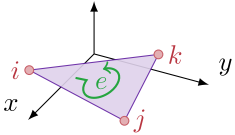
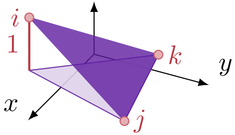
$$W_{\alpha_{1i}} = \begin{vmatrix} 1 & x_i & y_i \\ 0 & x_j & y_j \\ 0 & x_k & y_k \end{vmatrix} = x_j y_k - x_k y_j$$



$N_i(x^e, y^e)$



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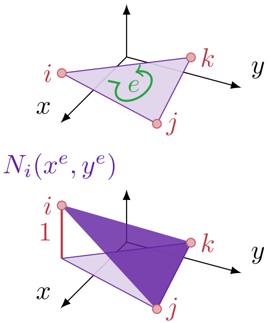

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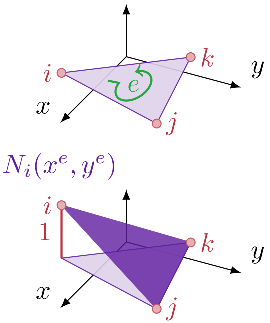

$$\alpha_{1i} = \frac{x_j y_k - x_k y_j}{2P_\triangle}$$

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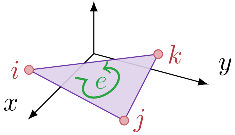
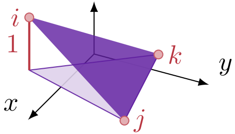


$N_i(x^e, y^e)$



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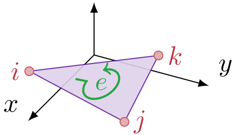

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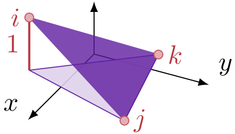

$$\alpha_{1i} = \frac{x_j y_k - x_k y_j}{2P_\triangle}$$

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Determination of shape functions for 3-node element

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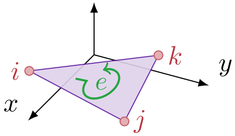
$$\boxed{\alpha_{3i} =} \frac{W_{\alpha_{3i}}}{W} = \boxed{\frac{x_k - x_j}{2P_\triangle}}$$



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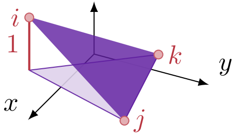

$$\boxed{\alpha_{1i} = \frac{x_j y_k - x_k y_j}{2P_\triangle}}$$

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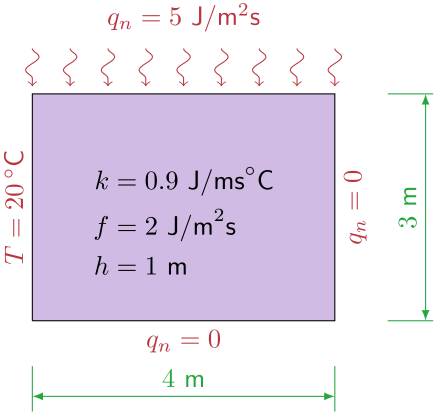
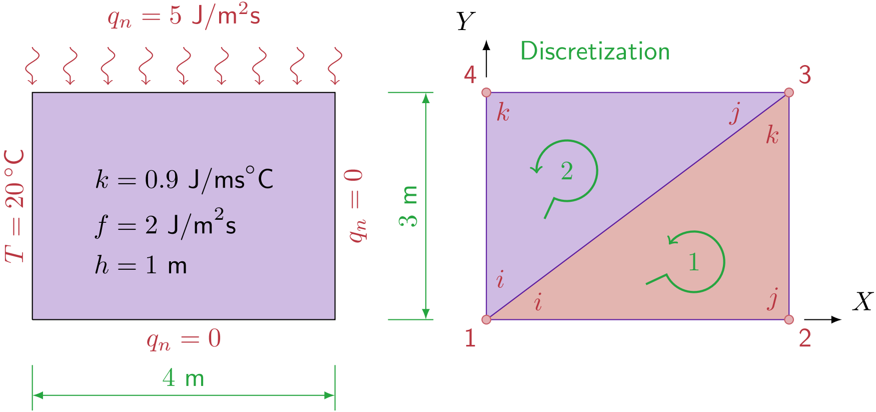
2D example of heat flow – 3-node elements



2D example of heat flow – 3-node elements



Heat flow in 2D

$$\int_A (\nabla w)^{\mathrm{T}} \mathbf{D} h \nabla T \mathrm{d}A = -\int_{\Gamma_\mathrm{q}} w h \widehat{q} \mathrm{d}\Gamma - \int_{\Gamma_\mathrm{T}} w h q_\mathrm{n} \mathrm{d}\Gamma + \int_A w h f \mathrm{d}A$$

$+$ boundary condition

$$T = \widehat{T} \quad \text{on } \Gamma_\mathrm{T}$$

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$$T = \mathbf{N}\boldsymbol{\Theta}, \qquad w = \mathbf{N}\mathbf{w} = \mathbf{w}^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}, \qquad \nabla T = \mathbf{B}\boldsymbol{\Theta}$$

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$$\nabla w = \mathbf{w}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}, \qquad \mathbf{D} = k\mathbf{I}, \qquad h = \mathrm{const}$$

$$\mathbf{w}^{\mathrm{T}} \int_A \mathbf{B}^{\mathrm{T}} k \mathbf{B} \mathrm{d}A \ \boldsymbol{\Theta} = -\mathbf{w}^{\mathrm{T}} \int_\Gamma \mathbf{N}^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma + \mathbf{w}^{\mathrm{T}} \int_A \mathbf{N}^{\mathrm{T}} f \mathrm{d}A, \quad \forall \mathbf{w}$$

Heat flow in 2D

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$$\mathbf{K} = \int_A \mathbf{B}^{\mathrm{T}} k \mathbf{B} \mathrm{d}A, \quad \mathbf{f} = \int_A \mathbf{N}^{\mathrm{T}} f \mathrm{d}A, \quad \mathbf{f}_b = -\int_\Gamma \mathbf{N}^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma$$

Heat flow in 2D

$$\int_A (\nabla w)^{\mathrm{T}} \mathbf{D} h \nabla T \mathrm{d}A = -\int_\Gamma w h q_{\mathrm{n}} \mathrm{d}\Gamma + \int_A w h f \mathrm{d}A$$

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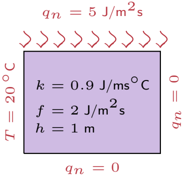
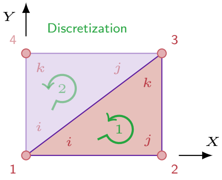
$$\mathbf{K\Theta} = \mathbf{f} + \mathbf{f}_b$$

2D example of heat flow – 3-node elements

$\mathbf{K}$ matrix – element 1

$$\mathbf{N}^1 = \left[\begin{array}{ccc} 1 - \frac{1}{4}x & \frac{1}{4}x - \frac{1}{3}y & \frac{1}{3}y \end{array} \right]$$





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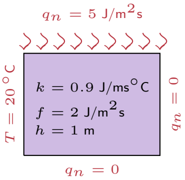
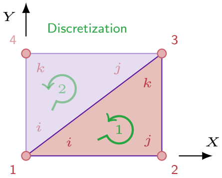
$$\mathbf{B}^1 = \nabla\mathbf{N} = \left[\begin{array}{rrr} -0.250 & 0.250 & 0.000 \\ 0.000 & -0.333 & 0.333 \end{array}\right]$$





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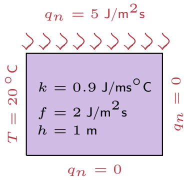
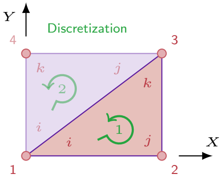
$$\mathbf{K}^1 = \int_{A^1} \mathbf{B}^{\mathrm{T}} k \mathbf{B} \mathrm{d}A = A^1 \mathbf{B}^{\mathrm{T}} k \mathbf{B}$$

$$= \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

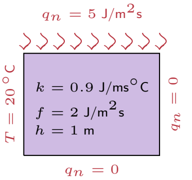


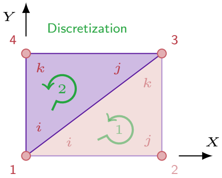


2D example of heat flow – 3-node elements

$\mathbf{K}$ matrix – element 2

$$\mathbf{N}^2 = \left[\begin{array}{ccc} 1-\frac{1}{3}y & \frac{1}{4}x & \frac{1}{3}y-\frac{1}{4}x \end{array}\right]$$





$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

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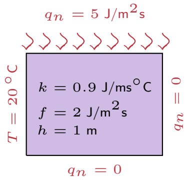
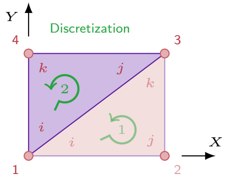
$$\mathbf{K}^1 = \left[\begin{array}{rrr} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{array}\right]$$





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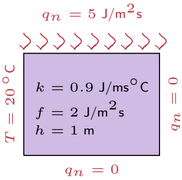
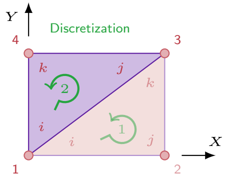
$$\mathbf{K}^2 = \int_{A^2} \mathbf{B}^{\mathrm{T}} k \mathbf{B} \mathrm{d}A = A^2 \mathbf{B}^{\mathrm{T}} k \mathbf{B}$$

$$= \left[\begin{array}{ccc} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{array}\right]$$

$$\mathbf{K}^1 = \left[\begin{array}{ccc} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{array}\right]$$





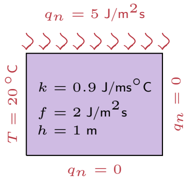
2D example of heat flow – 3-node elements

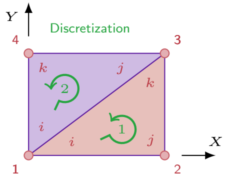
f vector – element 1 and 2 - $A^1 = A^2$

$$\mathbf{f}^e = \int_{A^e} \mathbf{N}^{\mathrm{T}} f \mathrm{d}A = \frac{f}{3} A^e \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$





2D example of heat flow – 3-node elements

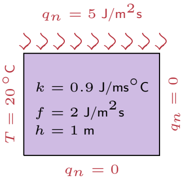
f_b vector – element 1

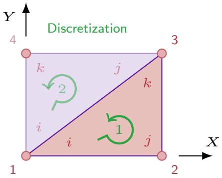
$$\mathbf{f}_b^1 = -\int_{\Gamma_{ij}^1} (\mathbf{N}^1)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma - \int_{\Gamma_{jk}^1} (\mathbf{N}^1)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma - \int_{\Gamma_{ki}^1} (\mathbf{N}^1)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma$$

$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$





2D example of heat flow – 3-node elements

$\mathbf{f}_b$ vector – element 1

$$\mathbf{f}_b^1 = -\int_{\Gamma_{ij}^1} (\mathbf{N}^1)^{\mathrm{T}} \overset{b.c.\,=\,0}{\boxed{q_{\mathrm{n}}}}\, \mathrm{d}\Gamma - \int_{\Gamma_{jk}^1} (\mathbf{N}^1)^{\mathrm{T}} \overset{b.c.\,=\,0}{\boxed{q_{\mathrm{n}}}}\, \mathrm{d}\Gamma$$

$$-\int_{\Gamma_{ki}^1} (\mathbf{N}^1)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma$$

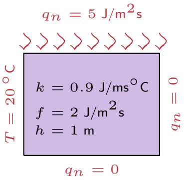
flow continuity along edge 1-3

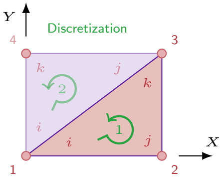
$q_{n}{}_{ki}^{1} = -q_{n}{}_{ij}^{2}$

$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$





2D example of heat flow – 3-node elements

f_b vector – element 1

$$\mathbf{f}_b^1 = -\int_{\Gamma_{ij}^1} (\mathbf{N}^1)^{\mathrm{T}} \underbrace{q_{\mathrm{n}}}_{\text{b.c.}=0} \, d\Gamma - \int_{\Gamma_{jk}^1} (\mathbf{N}^1)^{\mathrm{T}} \underbrace{q_{\mathrm{n}}}_{\text{b.c.}=0} \, d\Gamma$$

flow continuity along edge 1-3

$q_{n\,ki}^{\,1} = -q_{n\,ij}^{\,2}$

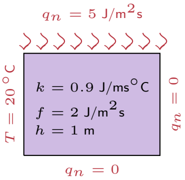
$$-\int_{\Gamma_{ki}^1} (\mathbf{N}^1)^{\mathrm{T}} q_{\mathrm{n}} d\Gamma$$

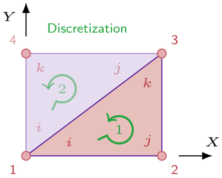
$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$





2D example of heat flow – 3-node elements

$\mathbf{f}_b$ vector – element 2

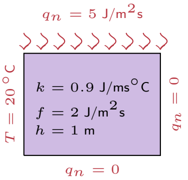
$$\mathbf{f}_b^2 = -\int_{\Gamma_{ij}^2} (\mathbf{N}^2)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma - \int_{\Gamma_{jk}^2} (\mathbf{N}^2)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma - \int_{\Gamma_{ki}^2} (\mathbf{N}^2)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma$$

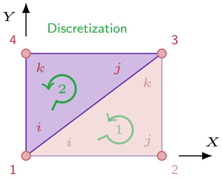
$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$





2D example of heat flow – 3-node elements

$\mathbf{f}_b$ vector – element 2

$$\mathbf{f}_b^2 = -\int_{\Gamma_{ij}^2} (\mathbf{N}^2)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma - \int_{\Gamma_{jk}^2} (\mathbf{N}^2)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma - \int_{\Gamma_{ki}^2} (\mathbf{N}^2)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma$$

flow continuity along edge 1-3

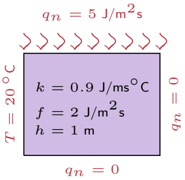
$$q_{n\,ki}^{\;1} = -q_{n\,ij}^{\;2}$$

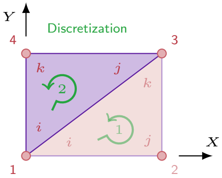
$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$





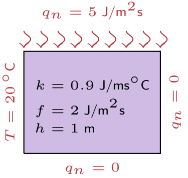
2D example of heat flow – 3-node elements

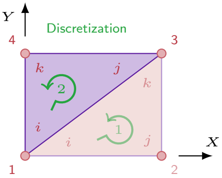
$\mathbf{f}_b$ vector – element 2

$$\mathbf{f}_b^2 = -\int_{\Gamma^2_{jk}} (\mathbf{N}^2)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma - \int_{\Gamma^2_{ki}} (\mathbf{N}^2)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma$$

$$-\int_{\Gamma^2_{jk}} (\mathbf{N}^2)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma = -\int_0^4 \left(\mathbf{N}^2(x, y=3)\right)^{\mathrm{T}} (-5)\mathrm{d}x$$

$$= \begin{bmatrix} 0 \\ 10 \\ 10 \end{bmatrix}$$





$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$\mathbf{f}_b$ vector – element 2

$$\mathbf{f}_b^2 = \begin{bmatrix} 0 \\ 10 \\ 10 \end{bmatrix} - \int_{\Gamma^2_{ki}} (\mathbf{N}^2)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma$$

$$-\int_{\Gamma^2_{ki}} (\mathbf{N}^2)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma = -\int_0^3 \left(\mathbf{N}^2(x{=}0,y)\right)^{\mathrm{T}} q_n \mathrm{d}x$$

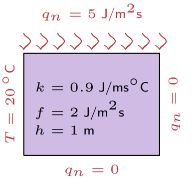
$$= \begin{bmatrix} f_{b1} \\ 0 \\ f_{b4} \end{bmatrix}$$

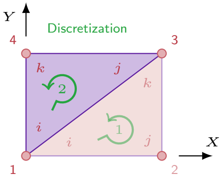
$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$



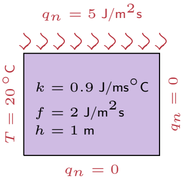


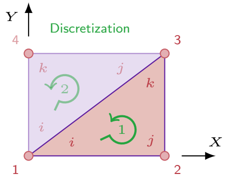
2D example of heat flow – 3-node elements

Assembly

$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 0.338 & -0.338 & 0.000 & 0.000 \\ -0.388 & 0.938 & -0.600 & 0.000 \\ 0.000 & -0.600 & 0.600 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \end{bmatrix}$$





$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} f_{b\,1} \\ 10 \\ f_{b\,4} + 10 \end{bmatrix}$$

2D example of heat flow – 3-node elements

Assembly

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

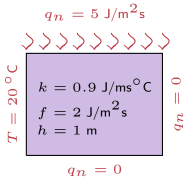
$$\mathbf{K} = \begin{bmatrix} 0.938 & -0.338 & 0.000 & -0.600 \\ -0.388 & 0.938 & -0.600 & 0.000 \\ 0.000 & -0.600 & 0.938 & -0.338 \\ -0.600 & 0.000 & -0.338 & 0.938 \end{bmatrix}$$

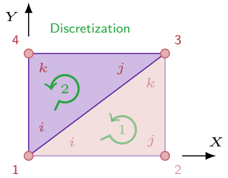
$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} f_{b\,1} \\ 10 \\ f_{b\,4} + 10 \end{bmatrix}$$



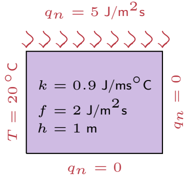


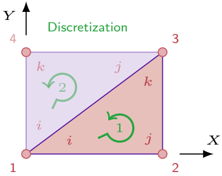
2D example of heat flow – 3-node elements

Assembly

$$\mathbf{f}^1 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix} \boxed{\begin{matrix} 4 \\ 4 \\ 4 \end{matrix}} \\ 0 \end{bmatrix}$$





$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} f_{b\,1} \\ 10 \\ f_{b\,4} + 10 \end{bmatrix}$$

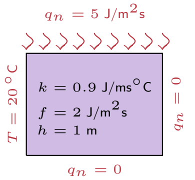
$$\mathbf{K} = \begin{bmatrix} 0.938 & -0.338 & 0.000 & -0.600 \\ -0.388 & 0.938 & -0.600 & 0.000 \\ 0.000 & -0.600 & 0.938 & -0.338 \\ -0.600 & 0.000 & -0.338 & 0.938 \end{bmatrix}$$

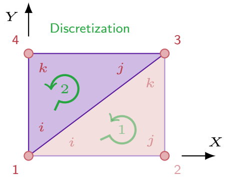
2D example of heat flow – 3-node elements

Assembly

$$\mathbf{f}^1 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \qquad \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix} 8 \\ 4 \\ 8 \\ 4 \end{bmatrix}$$





$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} f_{b\,1} \\ 10 \\ f_{b\,4} + 10 \end{bmatrix}$$

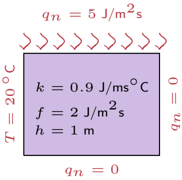
$$\mathbf{K} = \begin{bmatrix} 0.938 & -0.338 & 0.000 & -0.600 \\ -0.388 & 0.938 & -0.600 & 0.000 \\ 0.000 & -0.600 & 0.938 & -0.338 \\ -0.600 & 0.000 & -0.338 & 0.938 \end{bmatrix}$$

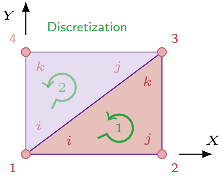
2D example of heat flow – 3-node elements

Assembly

$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{f}_b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$





$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} f_{b\,1} \\ 10 \\ f_{b\,4} + 10 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 0.938 & -0.338 & 0.000 & -0.600 \\ -0.388 & 0.938 & -0.600 & 0.000 \\ 0.000 & -0.600 & 0.938 & -0.338 \\ -0.600 & 0.000 & -0.338 & 0.938 \end{bmatrix}$$

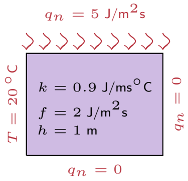
$$\mathbf{f} = \begin{bmatrix} 8 \\ 4 \\ 8 \\ 4 \end{bmatrix},$$

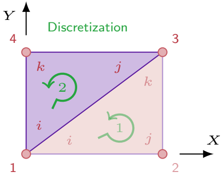
2D example of heat flow – 3-node elements

Assembly

$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{f}_b^2 = \begin{bmatrix} f_{b1} \\ 10 \\ f_{b4} + 10 \end{bmatrix}$$

$$\mathbf{f}_b = \begin{bmatrix} f_{b1} \\ 0 \\ 10 \\ f_{b4} + 10 \end{bmatrix}$$





$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} f_{b1} \\ 10 \\ f_{b4} + 10 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 0.938 & -0.338 & 0.000 & -0.600 \\ -0.388 & 0.938 & -0.600 & 0.000 \\ 0.000 & -0.600 & 0.938 & -0.338 \\ -0.600 & 0.000 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix} 8 \\ 4 \\ 8 \\ 4 \end{bmatrix},$$

2D example of heat flow – 3-node elements

FEM equation set: $\mathbf{K}\boldsymbol{\Theta} = \mathbf{f} + \mathbf{f}_b$

$$\begin{bmatrix} 0.938 & -0.338 & 0.000 & -0.600 \\ -0.388 & 0.938 & -0.600 & 0.000 \\ 0.000 & -0.600 & 0.938 & -0.338 \\ -0.600 & 0.000 & -0.338 & 0.938 \end{bmatrix} \begin{bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \\ \Theta_4 \end{bmatrix} =$$

$$= \begin{bmatrix} 8 \\ 4 \\ 8 \\ 4 \end{bmatrix} + \begin{bmatrix} f_{b1} \\ 0 \\ 10 \\ f_{b4} + 10 \end{bmatrix}$$

$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

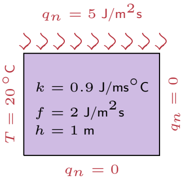
$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

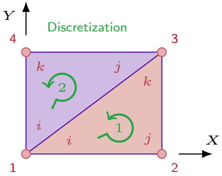
$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} f_{b1} \\ 10 \\ f_{b4} + 10 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 0.938 & -0.338 & 0.000 & -0.600 \\ -0.388 & 0.938 & -0.600 & 0.000 \\ 0.000 & -0.600 & 0.938 & -0.338 \\ -0.600 & 0.000 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix} 8 \\ 4 \\ 8 \\ 4 \end{bmatrix}, \quad \mathbf{f}_b = \begin{bmatrix} f_{b1} \\ 0 \\ 10 \\ f_{b4} + 10 \end{bmatrix}$$



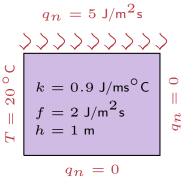


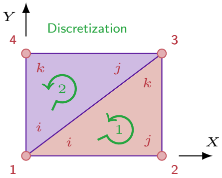
2D example of heat flow – 3-node elements

FEM equation set: $\mathbf{K}\boldsymbol{\Theta} = \mathbf{f} + \mathbf{f}_b$

$$\begin{bmatrix} 0.938 & -0.338 & 0.000 & -0.600 \\ -0.388 & 0.938 & -0.600 & 0.000 \\ 0.000 & -0.600 & 0.938 & -0.338 \\ -0.600 & 0.000 & -0.338 & 0.938 \end{bmatrix} \begin{bmatrix} 20 \\ \Theta_2 \\ \Theta_3 \\ 20 \end{bmatrix} =$$

$$= \begin{bmatrix} 8 \\ 4 \\ 8 \\ 4 \end{bmatrix} + \begin{bmatrix} f_{b1} \\ 0 \\ 10 \\ f_{b4} + 10 \end{bmatrix}$$





$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} f_{b1} \\ 10 \\ f_{b4} + 10 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 0.938 & -0.338 & 0.000 & -0.600 \\ -0.388 & 0.938 & -0.600 & 0.000 \\ 0.000 & -0.600 & 0.938 & -0.338 \\ -0.600 & 0.000 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix} 8 \\ 4 \\ 8 \\ 4 \end{bmatrix}, \quad \mathbf{f}_b = \begin{bmatrix} f_{b1} \\ 0 \\ 10 \\ f_{b4} + 10 \end{bmatrix}$$

2D example of heat flow – 3-node elements

FEM equation set: $\mathbf{K}\boldsymbol{\Theta} = \mathbf{f} + \mathbf{f}_b$

$$\begin{bmatrix} 0.938 & -0.338 & 0.000 & -0.600 \\ -0.388 & 0.938 & -0.600 & 0.000 \\ 0.000 & -0.600 & 0.938 & -0.338 \\ -0.600 & 0.000 & -0.338 & 0.938 \end{bmatrix} \begin{bmatrix} 20 \\ \Theta_2 \\ \Theta_3 \\ 20 \end{bmatrix} =$$

$$= \begin{bmatrix} 8 \\ 4 \\ 8 \\ 4 \end{bmatrix} + \begin{bmatrix} f_{b1} \\ 0 \\ 10 \\ f_{b4} + 10 \end{bmatrix}$$

Solution: $\Theta_2 = 48.040,\ \ \Theta_3 = 57.145,$
$f_{b1} = -17.463,\ \ f_{b4} = -26.537$

$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

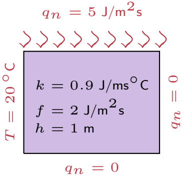
$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

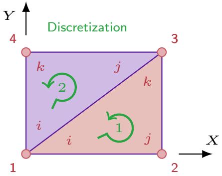
$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} f_{b1} \\ 10 \\ f_{b4} + 10 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 0.938 & -0.338 & 0.000 & -0.600 \\ -0.388 & 0.938 & -0.600 & 0.000 \\ 0.000 & -0.600 & 0.938 & -0.338 \\ -0.600 & 0.000 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix} 8 \\ 4 \\ 8 \\ 4 \end{bmatrix}, \quad \mathbf{f}_b = \begin{bmatrix} f_{b1} \\ 0 \\ 10 \\ f_{b4} + 10 \end{bmatrix}$$

$$\boldsymbol{\Theta} = \begin{bmatrix} 20 \\ 48.040 \\ 57.145 \\ 20 \end{bmatrix}$$

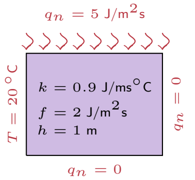


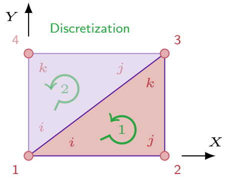


2D example of heat flow – 3-node elements

Heat flux vector – element 1

$$\mathbf{\Theta}^1 = \begin{bmatrix} 20 \\ 48.040 \\ 57.145 \end{bmatrix}$$





$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} f_{b\,1} \\ 10 \\ f_{b\,4} + 10 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 0.938 & -0.338 & 0.000 & -0.600 \\ -0.388 & 0.938 & -0.600 & 0.000 \\ 0.000 & -0.600 & 0.938 & -0.338 \\ -0.600 & 0.000 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix} 8 \\ 4 \\ 8 \\ 4 \end{bmatrix}, \quad \mathbf{f}_b = \begin{bmatrix} f_{b\,1} \\ 0 \\ 10 \\ f_{b\,4} + 10 \end{bmatrix}$$

$$\mathbf{\Theta} = \begin{bmatrix} 20 \\ 48.040 \\ 57.145 \\ 20 \end{bmatrix}$$

2D example of heat flow – 3-node elements

Heat flux vector – element 1

$$\mathbf{\Theta}^1 = \begin{bmatrix} 20 \\ 48.040 \\ 57.145 \end{bmatrix}$$

$$\mathbf{q}^1 = -k\mathbf{B}^1\mathbf{\Theta}^1$$

$$= -0.9 \begin{bmatrix} -0.250 & 0.250 & 0.000 \\ 0.000 & -0.333 & 0.333 \end{bmatrix} \begin{bmatrix} 20 \\ 48.040 \\ 57.145 \end{bmatrix}$$

$$= \begin{bmatrix} -6.309 \\ -2.732 \end{bmatrix}$$

$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

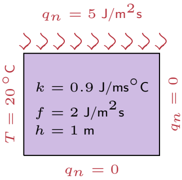
$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

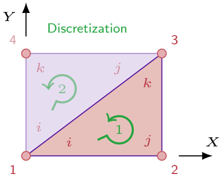
$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} f_{b\,1} \\ 10 \\ f_{b\,4} + 10 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 0.938 & -0.338 & 0.000 & -0.600 \\ -0.388 & 0.938 & -0.600 & 0.000 \\ 0.000 & -0.600 & 0.938 & -0.338 \\ -0.600 & 0.000 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix} 8 \\ 4 \\ 8 \\ 4 \end{bmatrix}, \quad \mathbf{f}_b = \begin{bmatrix} f_{b\,1} \\ 0 \\ 10 \\ f_{b\,4} + 10 \end{bmatrix}$$

$$\mathbf{\Theta} = \begin{bmatrix} 20 \\ 48.040 \\ 57.145 \\ 20 \end{bmatrix}$$

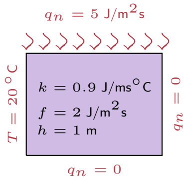


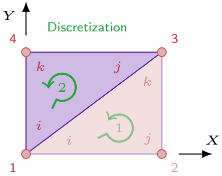


2D example of heat flow – 3-node elements

Heat flux vector – element 2

$$\boldsymbol{\Theta}^2 = \begin{bmatrix} 20 \\ 57.145 \\ 20 \end{bmatrix}$$





$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} f_{b\,1} \\ 10 \\ f_{b\,4} + 10 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 0.938 & -0.338 & 0.000 & -0.600 \\ -0.388 & 0.938 & -0.600 & 0.000 \\ 0.000 & -0.600 & 0.938 & -0.338 \\ -0.600 & 0.000 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix} 8 \\ 4 \\ 8 \\ 4 \end{bmatrix}, \quad \mathbf{f}_b = \begin{bmatrix} f_{b\,1} \\ 0 \\ 10 \\ f_{b\,4} + 10 \end{bmatrix}$$

$$\boldsymbol{\Theta} = \begin{bmatrix} 20 \\ 48.040 \\ 57.145 \\ 20 \end{bmatrix}$$

2D example of heat flow – 3-node elements

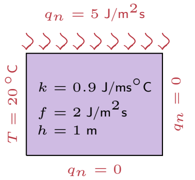
Heat flux vector – element 2

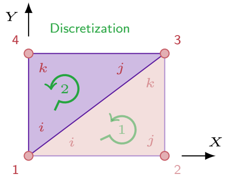
$$\boldsymbol{\Theta}^2 = \begin{bmatrix} 20 \\ 57.145 \\ 20 \end{bmatrix}$$

$$\mathbf{q}^2 = -k\mathbf{B}^2\boldsymbol{\Theta}^2$$

$$= -0.9 \begin{bmatrix} 0.000 & 0.250 & -0.250 \\ -0.333 & 0.000 & 0.333 \end{bmatrix} \begin{bmatrix} 20 \\ 57.145 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} -8.358 \\ 0.000 \end{bmatrix}$$





$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} f_{b\,1} \\ 10 \\ f_{b\,4} + 10 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 0.938 & -0.338 & 0.000 & -0.600 \\ -0.388 & 0.938 & -0.600 & 0.000 \\ 0.000 & -0.600 & 0.938 & -0.338 \\ -0.600 & 0.000 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix} 8 \\ 4 \\ 8 \\ 4 \end{bmatrix}, \quad \mathbf{f}_b = \begin{bmatrix} f_{b\,1} \\ 0 \\ 10 \\ f_{b\,4} + 10 \end{bmatrix}$$

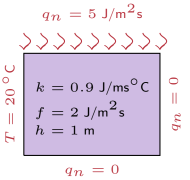
$$\boldsymbol{\Theta} = \begin{bmatrix} 20 \\ 48.040 \\ 57.145 \\ 20 \end{bmatrix}$$

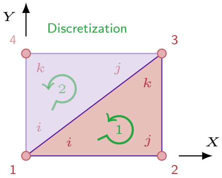
2D example of heat flow – 3-node elements

Computation of temperature at a point in el. 1

$$\mathbf{\Theta}^1 = \begin{bmatrix} 20 \\ 48.040 \\ 57.145 \end{bmatrix}$$

$$T^e(x^e, y^e) = \mathbf{N}^e(x^e, y^e)\mathbf{\Theta}^e$$





$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} f_{b\,1} \\ 10 \\ f_{b\,4} + 10 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 0.938 & -0.338 & 0.000 & -0.600 \\ -0.388 & 0.938 & -0.600 & 0.000 \\ 0.000 & -0.600 & 0.938 & -0.338 \\ -0.600 & 0.000 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix} 8 \\ 4 \\ 8 \\ 4 \end{bmatrix}, \quad \mathbf{f}_b = \begin{bmatrix} f_{b\,1} \\ 0 \\ 10 \\ f_{b\,4} + 10 \end{bmatrix}$$

$$\mathbf{\Theta} = \begin{bmatrix} 20 \\ 48.040 \\ 57.145 \\ 20 \end{bmatrix}$$

2D example of heat flow – 3-node elements

Computation of temperature at a point in el. 1

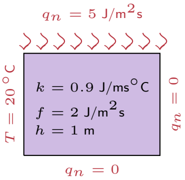
$$\boldsymbol{\Theta}^1 = \begin{bmatrix} 20 \\ 48.040 \\ 57.145 \end{bmatrix}$$

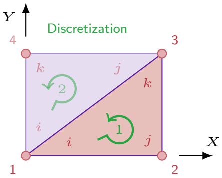
$$T^e(x^e, y^e) = \mathbf{N}^e(x^e, y^e)\boldsymbol{\Theta}^e$$

e.g. at centre of mass $(\frac{8}{3}, 1)$

$$T^1\left(\frac{8}{3}, 1\right) = \begin{bmatrix} 1-\frac{1}{4}\cdot\frac{8}{3} & \frac{1}{4}\cdot\frac{8}{3}-\frac{1}{3}\cdot 1 & \frac{1}{3}\cdot 1 \end{bmatrix} \begin{bmatrix} 20 \\ 48.040 \\ 57.145 \end{bmatrix}$$

$$= 41.728$$





$$\mathbf{K}^1 = \begin{bmatrix} 0.338 & -0.338 & 0.000 \\ -0.388 & 0.938 & -0.600 \\ 0.000 & -0.600 & 0.600 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.600 & 0.000 & -0.600 \\ 0.000 & 0.338 & -0.338 \\ -0.600 & -0.338 & 0.938 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

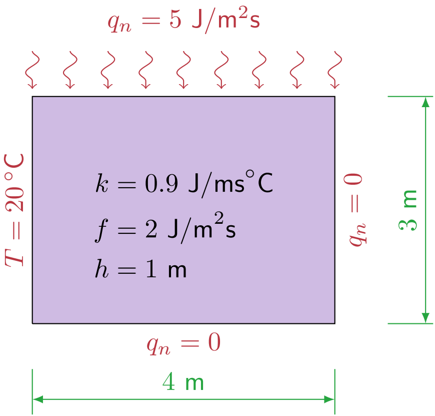
$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} f_{b\,1} \\ 10 \\ f_{b\,4} + 10 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 0.938 & -0.338 & 0.000 & -0.600 \\ -0.388 & 0.938 & -0.600 & 0.000 \\ 0.000 & -0.600 & 0.938 & -0.338 \\ -0.600 & 0.000 & -0.338 & 0.938 \end{bmatrix}$$

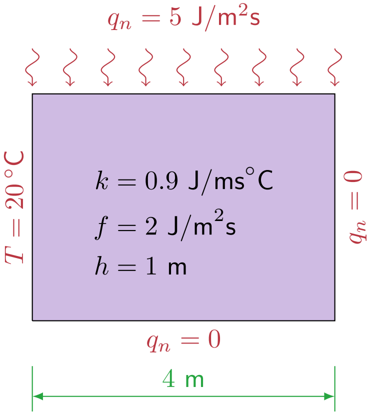
$$\mathbf{f} = \begin{bmatrix} 8 \\ 4 \\ 8 \\ 4 \end{bmatrix}, \quad \mathbf{f}_b = \begin{bmatrix} f_{b\,1} \\ 0 \\ 10 \\ f_{b\,4} + 10 \end{bmatrix}$$

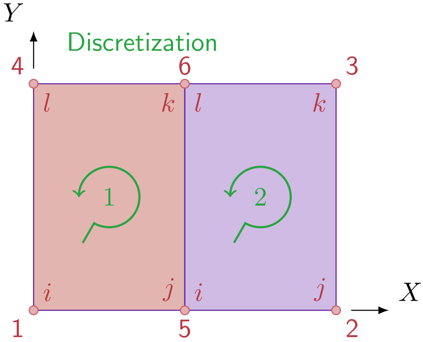
$$\boldsymbol{\Theta} = \begin{bmatrix} 20 \\ 48.040 \\ 57.145 \\ 20 \end{bmatrix}$$

2D example of heat flow – 4-node elements



2D example of heat flow – 4-node elements





Heat flow in 2D

$$\int_A (\nabla w)^{\mathrm{T}} \mathbf{D} h \nabla T \mathrm{d}A = -\int_{\Gamma_{\mathrm{q}}} w h \widehat{q} \mathrm{d}\Gamma - \int_{\Gamma_{\mathrm{T}}} w h q_{\mathrm{n}} \mathrm{d}\Gamma + \int_A w h f \mathrm{d}A$$

$+$ boundary condition

$$T = \widehat{T} \quad \text{on } \Gamma_{\mathrm{T}}$$

Heat flow in 2D

$$\int_A (\nabla w)^{\mathrm{T}} \mathbf{D} h \nabla T \mathrm{d}A = -\int_\Gamma w h q_{\mathrm{n}} \mathrm{d}\Gamma + \int_A w h f \mathrm{d}A$$

$+$ boundary condition

$$T = \widehat{T} \quad \text{on } \Gamma_{\mathrm{T}}$$

Heat flow in 2D

$$\int_A (\nabla w)^{\mathrm{T}} \mathbf{D} h \nabla T \mathrm{d}A = -\int_\Gamma w h q_{\mathrm{n}} \mathrm{d}\Gamma + \int_A w h f \mathrm{d}A$$

+ boundary condition

$$T = \widehat{T} \quad \text{on } \Gamma_{\mathrm{T}}$$

$$T = \mathbf{N\Theta}, \qquad w = \mathbf{Nw} = \mathbf{w}^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}, \qquad \nabla T = \mathbf{B\Theta}$$

$$\nabla w = \mathbf{w}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}, \qquad \mathbf{D} = k\mathbf{I}, \qquad h = \text{const}$$

Heat flow in 2D

$$\int_A (\nabla w)^{\mathrm{T}} \mathbf{D} h \nabla T \mathrm{d}A = -\int_\Gamma w h q_{\mathrm{n}} \mathrm{d}\Gamma + \int_A w h f \mathrm{d}A$$

+ boundary condition

$$T = \widehat{T} \quad \text{on } \Gamma_{\mathrm{T}}$$

$$T = \mathbf{N\Theta}, \qquad w = \mathbf{Nw} = \mathbf{w}^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}, \qquad \nabla T = \mathbf{B\Theta}$$

$$\nabla w = \mathbf{w}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}, \qquad \mathbf{D} = k\mathbf{I}, \qquad h = \mathrm{const}$$

$$\mathbf{w}^{\mathrm{T}} \int_A \mathbf{B}^{\mathrm{T}} k \mathbf{B} \mathrm{d}A \; \mathbf{\Theta} = -\mathbf{w}^{\mathrm{T}} \int_\Gamma \mathbf{N}^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma + \mathbf{w}^{\mathrm{T}} \int_A \mathbf{N}^{\mathrm{T}} f \mathrm{d}A, \quad \forall \mathbf{w}$$

Heat flow in 2D

$$\int_A (\nabla w)^{\mathrm{T}} \mathbf{D} h \nabla T \mathrm{d}A = -\int_\Gamma w h q_{\mathrm{n}} \mathrm{d}\Gamma + \int_A w h f \mathrm{d}A$$

+ boundary condition

$$T = \widehat{T} \quad \text{on } \Gamma_{\mathrm{T}}$$

$$T = \mathbf{N}\boldsymbol{\Theta}, \qquad w = \mathbf{N}\mathbf{w} = \mathbf{w}^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}, \qquad \nabla T = \mathbf{B}\boldsymbol{\Theta}$$

$$\nabla w = \mathbf{w}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}, \qquad \mathbf{D} = k\mathbf{I}, \qquad h = \mathrm{const}$$

$$\int_A \mathbf{B}^{\mathrm{T}} k \mathbf{B} \mathrm{d}A \; \boldsymbol{\Theta} = -\int_\Gamma \mathbf{N}^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma + \int_A \mathbf{N}^{\mathrm{T}} f \mathrm{d}A$$

Heat flow in 2D

$$\int_A (\nabla w)^{\mathrm{T}} \mathbf{D} h \nabla T \mathrm{d}A = -\int_\Gamma w h q_{\mathrm{n}} \mathrm{d}\Gamma + \int_A w h f \mathrm{d}A$$

+ boundary condition

$$T = \widehat{T} \quad \text{on } \Gamma_{\mathrm{T}}$$

$$T = \mathbf{N}\boldsymbol{\Theta}, \qquad w = \mathbf{N}\mathbf{w} = \mathbf{w}^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}, \qquad \nabla T = \mathbf{B}\boldsymbol{\Theta}$$

$$\nabla w = \mathbf{w}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}, \qquad \mathbf{D} = k\mathbf{I}, \qquad h = \mathrm{const}$$

$$\int_A \mathbf{B}^{\mathrm{T}} k \mathbf{B} \mathrm{d}A \; \boldsymbol{\Theta} = -\int_\Gamma \mathbf{N}^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma + \int_A \mathbf{N}^{\mathrm{T}} f \mathrm{d}A$$

$$\mathbf{K} = \int_A \mathbf{B}^{\mathrm{T}} k \mathbf{B} \mathrm{d}A, \quad \mathbf{f} = \int_A \mathbf{N}^{\mathrm{T}} f \mathrm{d}A, \quad \mathbf{f}_b = -\int_\Gamma \mathbf{N}^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma$$

Heat flow in 2D

$$\int_A (\nabla w)^{\mathrm{T}} \mathbf{D} h \nabla T \mathrm{d}A = -\int_\Gamma w h q_{\mathrm{n}} \mathrm{d}\Gamma + \int_A w h f \mathrm{d}A$$

+ boundary condition

$$T = \widehat{T} \quad \text{on } \Gamma_{\mathrm{T}}$$

$$T = \mathbf{N}\boldsymbol{\Theta}, \qquad w = \mathbf{N}\mathbf{w} = \mathbf{w}^{\mathrm{T}}\mathbf{N}^{\mathrm{T}}, \qquad \nabla T = \mathbf{B}\boldsymbol{\Theta}$$

$$\nabla w = \mathbf{w}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}, \qquad \mathbf{D} = k\mathbf{I}, \qquad h = \mathrm{const}$$

$$\int_A \mathbf{B}^{\mathrm{T}} k \mathbf{B} \mathrm{d}A \ \boldsymbol{\Theta} = -\int_\Gamma \mathbf{N}^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma + \int_A \mathbf{N}^{\mathrm{T}} f \mathrm{d}A$$

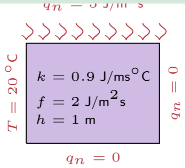
$$\mathbf{K} = \int_A \mathbf{B}^{\mathrm{T}} k \mathbf{B} \mathrm{d}A, \quad \mathbf{f} = \int_A \mathbf{N}^{\mathrm{T}} f \mathrm{d}A, \quad \mathbf{f}_b = -\int_\Gamma \mathbf{N}^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma$$

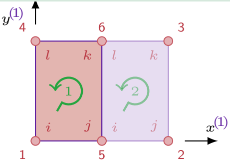
$$\mathbf{K}\boldsymbol{\Theta} = \mathbf{f} + \mathbf{f}_b$$

2D example of heat flow – 4-node elements

$\mathbf{K}$ matrix – element 1

$$\mathbf{N}^1 = \left[\begin{array}{cccc} \frac{(x^{(1)}-2)(y^{(1)}-3)}{2\cdot 3} & \frac{x^{(1)}(y^{(1)}-3)}{-2\cdot 3} & \frac{x^{(1)}y^{(1)}}{2\cdot 3} & \frac{(x^{(1)}-2)y^{(1)}}{-2\cdot 3} \end{array} \right]$$



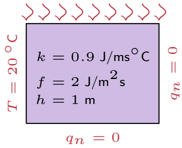


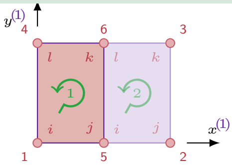
2D example of heat flow – 4-node elements

$\mathbf{K}$ matrix – element 1

$$\mathbf{N}^1 = \left[\begin{array}{cccc} \frac{(x^{(1)}-2)(y^{(1)}-3)}{2\cdot 3} & \frac{x^{(1)}(y^{(1)}-3)}{-2\cdot 3} & \frac{x^{(1)}y^{(1)}}{2\cdot 3} & \frac{(x^{(1)}-2)y^{(1)}}{-2\cdot 3} \end{array}\right]$$

$$\mathbf{B}^1 = \nabla\mathbf{N}^1 = \left[\begin{array}{cccc} \frac{y^{(1)}-3}{6} & \frac{y^{(1)}-3}{-6} & \frac{y^{(1)}}{6} & \frac{y^{(1)}}{-6} \\ \frac{x^{(1)}-2}{6} & \frac{x^{(1)}}{-6} & \frac{x^{(1)}}{6} & \frac{x^{(1)}-2}{-6} \end{array}\right]$$





2D example of heat flow – 4-node elements

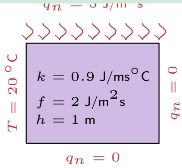
$\mathbf{K}$ matrix – element 1

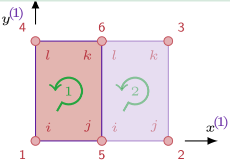
$$\mathbf{N}^1 = \left[\ \frac{(x^{(1)}-2)(y^{(1)}-3)}{2\cdot 3} \quad \frac{x^{(1)}(y^{(1)}-3)}{-2\cdot 3} \quad \frac{x^{(1)}y^{(1)}}{2\cdot 3} \quad \frac{(x^{(1)}-2)y^{(1)}}{-2\cdot 3}\ \right]$$

$$\mathbf{B}^1 = \nabla\mathbf{N}^1 = \begin{bmatrix} \frac{y^{(1)}-3}{6} & \frac{y^{(1)}-3}{-6} & \frac{y^{(1)}}{6} & \frac{y^{(1)}}{-6} \\ \frac{x^{(1)}-2}{6} & \frac{x^{(1)}}{-6} & \frac{x^{(1)}}{6} & \frac{x^{(1)}-2}{-6} \end{bmatrix}$$

$$\mathbf{K}^1 = \int_{A^1} \mathbf{B}^{\mathrm{T}} k \mathbf{B}\,\mathrm{d}A = \int_0^3\int_0^2 \mathbf{B}^{\mathrm{T}} k \mathbf{B}\,\mathrm{d}x^{(1)}\,\mathrm{d}y^{(1)} =$$

$$= \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$



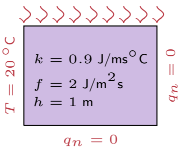


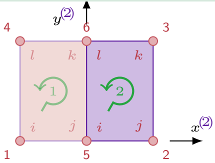
2D example of heat flow – 4-node elements

$\mathbf{K}$ matrix – element 2

$$\mathbf{N}^2 = \left[\begin{array}{cccc} \frac{(x^{(2)}-2)(y^{(2)}-3)}{2\cdot 3} & \frac{x^{(2)}(y^{(2)}-3)}{-2\cdot 3} & \frac{x^{(2)}y^{(2)}}{2\cdot 3} & \frac{(x^{(2)}-2)y^{(2)}}{-2\cdot 3} \end{array} \right]$$

$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$



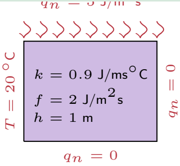


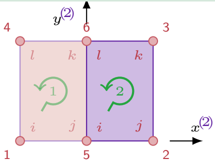
2D example of heat flow – 4-node elements

$\mathbf{K}$ matrix – element 2

$$\mathbf{N}^2 = \left[\begin{array}{cccc} \frac{(x^{(2)}-2)(y^{(2)}-3)}{2\cdot 3} & \frac{x^{(2)}(y^{(2)}-3)}{-2\cdot 3} & \frac{x^{(2)}y^{(2)}}{2\cdot 3} & \frac{(x^{(2)}-2)y^{(2)}}{-2\cdot 3} \end{array}\right]$$

$$\mathbf{B}^2 = \nabla\mathbf{N}^2 = \left[\begin{array}{cccc} \frac{y^{(2)}-3}{6} & \frac{y^{(2)}-3}{-6} & \frac{y^{(2)}}{6} & \frac{y^{(2)}}{-6} \\ \frac{x^{(2)}-2}{6} & \frac{x^{(2)}}{-6} & \frac{x^{(2)}}{6} & \frac{x^{(2)}-2}{-6} \end{array}\right]$$





$$\mathbf{K}^1 = \left[\begin{array}{rrrr} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{array}\right]$$

2D example of heat flow – 4-node elements

$\mathbf{K}$ matrix – element 2

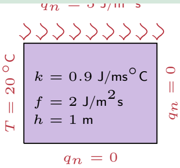
$$\mathbf{N}^2 = \left[\ \frac{(x^{(2)}-2)(y^{(2)}-3)}{2\cdot 3} \quad \frac{x^{(2)}(y^{(2)}-3)}{-2\cdot 3} \quad \frac{x^{(2)}y^{(2)}}{2\cdot 3} \quad \frac{(x^{(2)}-2)y^{(2)}}{-2\cdot 3}\ \right]$$

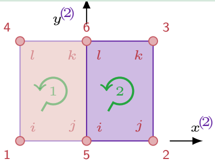
$$\mathbf{B}^2 = \nabla \mathbf{N}^2 = \begin{bmatrix} \frac{y^{(2)}-3}{6} & \frac{y^{(2)}-3}{-6} & \frac{y^{(2)}}{6} & \frac{y^{(2)}}{-6} \\ \frac{x^{(2)}-2}{6} & \frac{x^{(2)}}{-6} & \frac{x^{(2)}}{6} & \frac{x^{(2)}-2}{-6} \end{bmatrix}$$

$$\mathbf{K}^2 = \int_{A^2} \mathbf{B}^{\mathrm{T}} k \mathbf{B} \mathrm{d}A = \int_0^3 \int_0^2 \mathbf{B}^{\mathrm{T}} k \mathbf{B} \mathrm{d}x^{(2)} \mathrm{d}y^{(2)} =$$

$$= \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$





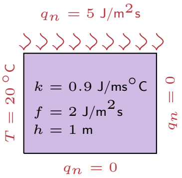
2D example of heat flow – 4-node elements

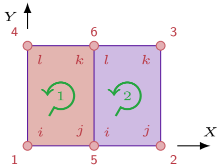
f vector – element 1 and 2 - $A^1 = A^2$

$$\mathbf{f}^e = \int_{A^e} \mathbf{N}^{\mathrm{T}} f \mathrm{d}A = \frac{f}{4} A^e \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

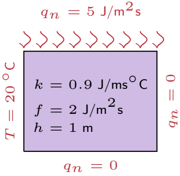


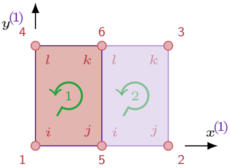


2D example of heat flow – 4-node elements

$\mathbf{f}_b$ vector – element 1

$$\mathbf{f}_b^1 = -\int_{\Gamma_{ij}^1} (\mathbf{N}^1)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma - \int_{\Gamma_{jk}^1} (\mathbf{N}^1)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma - \int_{\Gamma_{kl}^1} (\mathbf{N}^1)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma - \int_{\Gamma_{li}^1} (\mathbf{N}^1)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma$$





$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

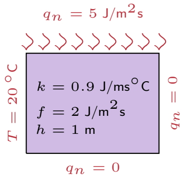
$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

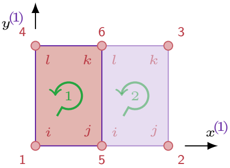
2D example of heat flow – 4-node elements

$\mathbf{f}_b$ vector – element 1

$$\mathbf{f}_b^1 = -\int_{\Gamma^1_{ij}} (\mathbf{N}^1)^{\mathrm{T}} \underbrace{\boxed{q_{\mathrm{n}}}}_{b.c.\,=\,0} \mathrm{d}\Gamma - \int_{\Gamma^1_{jk}} (\mathbf{N}^1)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma - \int_{\Gamma^1_{kl}} (\mathbf{N}^1)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma - \int_{\Gamma^1_{li}} (\mathbf{N}^1)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma$$

flow continuity
along edge 5-6
$q_{n}{}^{1}_{jk} = -q_{n}{}^{2}_{li}$





$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

2D example of heat flow – 4-node elements

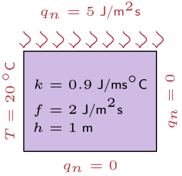
$\mathbf{f}_b$ vector – element 1

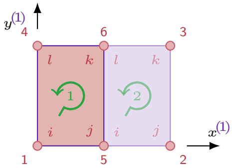
$$\mathbf{f}_b^1 = -\int_{\Gamma_{kl}^1} (\mathbf{N}^1)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma - \int_{\Gamma_{li}^1} (\mathbf{N}^1)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma$$

$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$





2D example of heat flow – 4-node elements

$\mathbf{f}_b$ vector – element 1

$$\mathbf{f}_b^1 = -\int_{\Gamma^1_{kl}} (\mathbf{N}^1)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma - \int_{\Gamma^1_{li}} (\mathbf{N}^1)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma$$

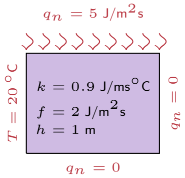
$$-\int_{\Gamma^1_{kl}} (\mathbf{N}^1)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma = -\int_0^2 \left(\mathbf{N}^1(x^{(1)}, y^{(1)} = 3)\right)^{\mathrm{T}} (-5) \mathrm{d}x^{(1)}$$

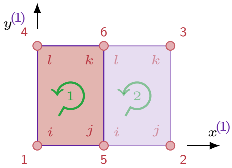
$$= \begin{bmatrix} 0 \\ 0 \\ 5 \\ 5 \end{bmatrix}$$

$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$





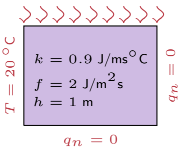
2D example of heat flow – 4-node elements

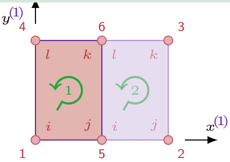
$\mathbf{f}_b$ vector – element 1

$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 5 \end{bmatrix} - \int_{\Gamma_{li}^1} (\mathbf{N}^1)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma$$

$$-\int_{\Gamma_{li}^1} (\mathbf{N}^1)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma = -\int_0^3 \left(\mathbf{N}^1(x^{(1)}=0, y^{(1)})\right)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}y^{(1)}$$

$$= \begin{bmatrix} f_{b1} \\ 0 \\ 0 \\ f_{b4} \end{bmatrix}$$





$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

2D example of heat flow – 4-node elements

$\mathbf{f}_b$ vector – element 1

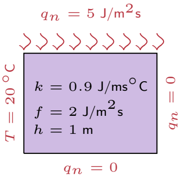
$$\mathbf{f}_b^1 = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 5 \end{bmatrix} + \begin{bmatrix} f_{b1} \\ 0 \\ 0 \\ f_{b4} \end{bmatrix}$$

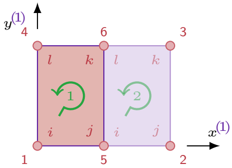
$$\mathbf{f}_b^1 = \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{bmatrix}$$

$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$





2D example of heat flow – 4-node elements

f_b vector – element 2

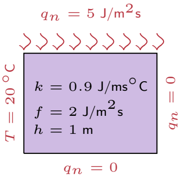
$$\mathbf{f}_b^2 = -\int_{\Gamma_{ij}^2} (\mathbf{N}^2)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma - \int_{\Gamma_{jk}^2} (\mathbf{N}^2)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma - \int_{\Gamma_{kl}^2} (\mathbf{N}^2)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma - \int_{\Gamma_{li}^2} (\mathbf{N}^2)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma$$

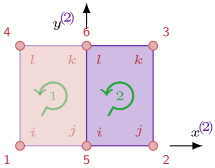
$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{bmatrix},$$





2D example of heat flow – 4-node elements

$\mathbf{f}_b$ vector – element 2

$$\mathbf{f}_b^2 = -\int_{\Gamma_{ij}^2} (\mathbf{N}^2)^{\mathrm{T}} \overset{b.c.\,=\,0}{q_{\mathrm{n}}} \, \mathrm{d}\Gamma - \int_{\Gamma_{jk}^2} (\mathbf{N}^2)^{\mathrm{T}} \overset{b.c.\,=\,0}{q_{\mathrm{n}}} \, \mathrm{d}\Gamma$$

$$- \int_{\Gamma_{kl}^2} (\mathbf{N}^2)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma - \int_{\Gamma_{li}^2} (\mathbf{N}^2)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma$$

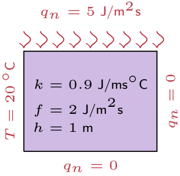
flow continuity
along edge 5-6
$q_{n\,jk}^{\,1} = -q_{n\,li}^{\,2}$

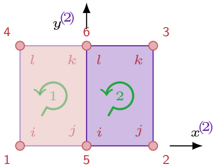
$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{bmatrix},$$





2D example of heat flow – 4-node elements

$\mathbf{f}_b$ vector – element 2

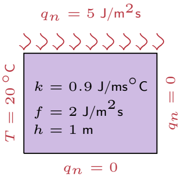
$$\mathbf{f}_b^2 = -\int_{\Gamma_{kl}^2} (\mathbf{N}^2)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma$$

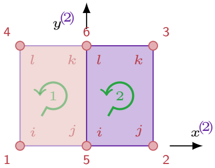
$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{bmatrix},$$





2D example of heat flow – 4-node elements

$\mathbf{f}_b$ vector – element 2

$$\mathbf{f}_b^2 = -\int_{\Gamma_{kl}^2} (\mathbf{N}^2)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma$$

$$-\int_{\Gamma_{kl}^2} (\mathbf{N}^2)^{\mathrm{T}} q_{\mathrm{n}} \mathrm{d}\Gamma = -\int_0^2 \left(\mathbf{N}^1(x^{(2)}, y^{(2)} = 3)\right)^{\mathrm{T}} (-5) \mathrm{d}x^{(2)}$$

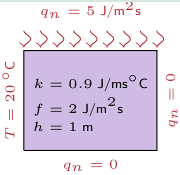
$$= \begin{bmatrix} 0 \\ 0 \\ 5 \\ 5 \end{bmatrix}$$

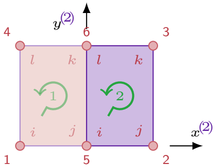
$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{bmatrix},$$





2D example of heat flow – 4-node elements

$\mathbf{f}_b$ vector – element 2

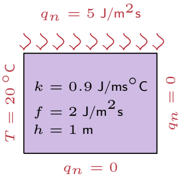
$$\mathbf{f}_b^2 = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 5 \end{bmatrix}$$

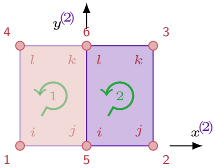
$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{bmatrix},$$



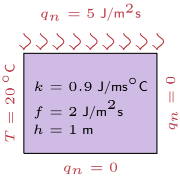


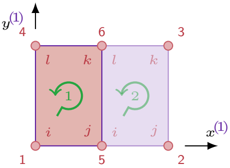
2D example of heat flow – 4-node elements

Assembly

$$\begin{array}{cc} \mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix} & \begin{array}{c} 1 \\ 5 \\ 6 \\ 4 \end{array} \\ \begin{array}{cccc} \quad 1 & \quad 5 & \quad 6 & \quad 4 \end{array} & \end{array}$$

$$\mathbf{K} = \begin{bmatrix} 0.650 & 0.000 & 0.000 & 0.025 & -0.350 & -0.325 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.025 & 0.000 & 0.000 & 0.650 & -0.325 & -0.350 \\ -0.350 & 0.000 & 0.000 & -0.325 & 0.650 & 0.025 \\ -0.325 & 0.000 & 0.000 & -0.350 & 0.025 & 0.650 \end{bmatrix}$$





$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 5 \end{bmatrix}$$

2D example of heat flow – 4-node elements

Assembly

$$
\begin{array}{cc}
\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix} & \begin{matrix} 5 \\ 2 \\ 3 \\ 6 \end{matrix} \\
\begin{matrix} \quad\quad\quad 5 & \quad\quad 2 & \quad\quad 3 & \quad\quad 6 \end{matrix} &
\end{array}
$$

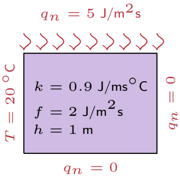
$$
\mathbf{K} = \begin{bmatrix}
0.650 & 0.000 & 0.000 & 0.025 & -0.350 & -0.325 \\
0.000 & 0.650 & 0.025 & 0.000 & -0.350 & -0.325 \\
0.000 & 0.025 & 0.650 & 0.000 & -0.325 & -0.350 \\
0.025 & 0.000 & 0.000 & 0.650 & -0.325 & -0.350 \\
-0.350 & -0.350 & -0.325 & -0.325 & 1.300 & 0.050 \\
-0.325 & -0.325 & -0.350 & -0.350 & 0.050 & 1.300
\end{bmatrix}
$$

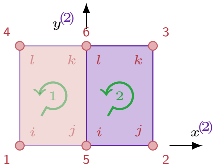
$$
\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}
$$

$$
\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}
$$

$$
\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}
$$

$$
\mathbf{f}_b^1 = \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 5 \end{bmatrix}
$$





2D example of heat flow – 4-node elements

Assembly

$$\mathbf{f}^1 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} \begin{matrix} 1 \\ 5 \\ 6 \\ 4 \end{matrix}$$

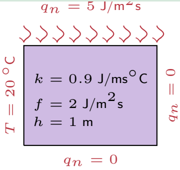
$$\mathbf{f} = \begin{bmatrix} \boxed{3} \\ 0 \\ 0 \\ \boxed{3} \\ \boxed{3} \\ \boxed{3} \end{bmatrix}$$

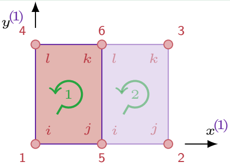
$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 5 \end{bmatrix}$$





2D example of heat flow – 4-node elements

Assembly

$$\mathbf{f}^1 = \begin{array}{c} \left[\begin{array}{c} 3 \\ 3 \\ 3 \\ 3 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 5 \\ 6 \\ 4 \end{array} \qquad \mathbf{f}^2 = \begin{array}{c} \left[\begin{array}{c} 3 \\ 3 \\ 3 \\ 3 \end{array}\right] \end{array} \begin{array}{c} 5 \\ 2 \\ 3 \\ 6 \end{array}$$

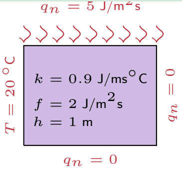
$$\mathbf{f} = \left[\begin{array}{c} 3 \\ 3 \\ 3 \\ 3 \\ 6 \\ 6 \end{array}\right]$$

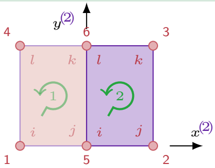
$$\mathbf{K}^1 = \left[\begin{array}{rrrr} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{array}\right]$$

$$\mathbf{K}^2 = \left[\begin{array}{rrrr} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{array}\right]$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \left[\begin{array}{c} 3 \\ 3 \\ 3 \\ 3 \end{array}\right]$$

$$\mathbf{f}_b^1 = \left[\begin{array}{c} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{array}\right], \quad \mathbf{f}_b^2 = \left[\begin{array}{c} 0 \\ 0 \\ 5 \\ 5 \end{array}\right]$$





Assembly

$$\mathbf{f}_b^1 = \begin{array}{c} \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 5 \\ 6 \\ 4 \end{array}$$

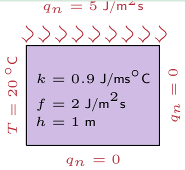
$$\mathbf{f}_b = \begin{bmatrix} \boxed{f_{b1}} \\ 0 \\ 0 \\ \boxed{5 + f_{b4}} \\ \boxed{0} \\ \boxed{5} \end{bmatrix}$$

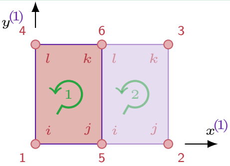
$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 5 \end{bmatrix}$$



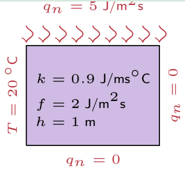


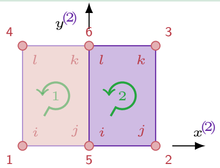
2D example of heat flow – 4-node elements

Assembly

$$\mathbf{f}_b^1 = \begin{array}{cc} \left[\begin{array}{c} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{array}\right] & \begin{array}{c} 1 \\ 5 \\ 6 \\ 4 \end{array} \end{array} \qquad \mathbf{f}_b^2 = \begin{array}{cc} \left[\begin{array}{c} 0 \\ 0 \\ 5 \\ 5 \end{array}\right] & \begin{array}{c} 5 \\ 2 \\ 3 \\ 6 \end{array} \end{array}$$

$$\mathbf{f}_b = \left[\begin{array}{c} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \\ 0 \\ 10 \end{array}\right]$$





$$\mathbf{K}^1 = \left[\begin{array}{rrrr} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{array}\right]$$

$$\mathbf{K}^2 = \left[\begin{array}{rrrr} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{array}\right]$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \left[\begin{array}{c} 3 \\ 3 \\ 3 \\ 3 \end{array}\right]$$

$$\mathbf{f}_b^1 = \left[\begin{array}{c} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{array}\right], \quad \mathbf{f}_b^2 = \left[\begin{array}{c} 0 \\ 0 \\ 5 \\ 5 \end{array}\right]$$

2D example of heat flow – 4-node elements

FEM equation set: $\mathbf{K}\boldsymbol{\Theta} = \mathbf{f} + \mathbf{f}_b$

$$
\begin{bmatrix}
0.650 & 0.000 & 0.000 & 0.025 & -0.350 & -0.325 \\
0.000 & 0.650 & 0.025 & 0.000 & -0.350 & -0.325 \\
0.000 & 0.025 & 0.650 & 0.000 & -0.325 & -0.350 \\
0.025 & 0.000 & 0.000 & 0.650 & -0.325 & -0.350 \\
-0.350 & -0.350 & -0.325 & -0.325 & 1.300 & 0.050 \\
-0.325 & -0.325 & -0.350 & -0.350 & 0.050 & 1.300
\end{bmatrix}
\begin{bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \\ \Theta_4 \\ \Theta_5 \\ \Theta_6 \end{bmatrix} =
$$

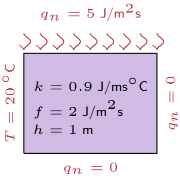
$$
= \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 6 \\ 6 \end{bmatrix} + \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \\ 0 \\ 10 \end{bmatrix}
$$

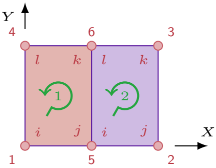
$$
\mathbf{K}^1 = \begin{bmatrix}
0.650 & -0.350 & -0.325 & 0.025 \\
-0.350 & 0.650 & 0.025 & -0.325 \\
-0.325 & 0.025 & 0.650 & -0.350 \\
0.025 & -0.325 & -0.350 & 0.650
\end{bmatrix}
$$

$$
\mathbf{K}^2 = \begin{bmatrix}
0.650 & -0.350 & -0.325 & 0.025 \\
-0.350 & 0.650 & 0.025 & -0.325 \\
-0.325 & 0.025 & 0.650 & -0.350 \\
0.025 & -0.325 & -0.350 & 0.650
\end{bmatrix}
$$

$$
\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}
$$

$$
\mathbf{f}_b^1 = \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 5 \end{bmatrix}
$$





2D example of heat flow – 4-node elements

FEM equation set: $\mathbf{K}\boldsymbol{\Theta} = \mathbf{f} + \mathbf{f}_b$

$$
\begin{bmatrix}
0.650 & 0.000 & 0.000 & 0.025 & -0.350 & -0.325 \\
0.000 & 0.650 & 0.025 & 0.000 & -0.350 & -0.325 \\
0.000 & 0.025 & 0.650 & 0.000 & -0.325 & -0.350 \\
0.025 & 0.000 & 0.000 & 0.650 & -0.325 & -0.350 \\
-0.350 & -0.350 & -0.325 & -0.325 & 1.300 & 0.050 \\
-0.325 & -0.325 & -0.350 & -0.350 & 0.050 & 1.300
\end{bmatrix}
\begin{bmatrix} 20 \\ \Theta_2 \\ \Theta_3 \\ 20 \\ \Theta_5 \\ \Theta_6 \end{bmatrix} =
$$

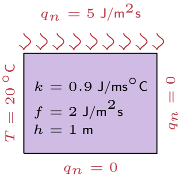
$$
= \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 6 \\ 6 \end{bmatrix} + \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \\ 0 \\ 10 \end{bmatrix}
$$

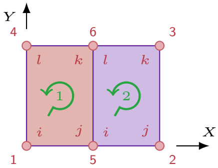
$$
\mathbf{K}^1 = \begin{bmatrix}
0.650 & -0.350 & -0.325 & 0.025 \\
-0.350 & 0.650 & 0.025 & -0.325 \\
-0.325 & 0.025 & 0.650 & -0.350 \\
0.025 & -0.325 & -0.350 & 0.650
\end{bmatrix}
$$

$$
\mathbf{K}^2 = \begin{bmatrix}
0.650 & -0.350 & -0.325 & 0.025 \\
-0.350 & 0.650 & 0.025 & -0.325 \\
-0.325 & 0.025 & 0.650 & -0.350 \\
0.025 & -0.325 & -0.350 & 0.650
\end{bmatrix}
$$

$$
\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}
$$

$$
\mathbf{f}_b^1 = \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 5 \end{bmatrix}
$$





2D example of heat flow – 4-node elements

FEM equation set: $\mathbf{K}\boldsymbol{\Theta} = \mathbf{f} + \mathbf{f}_b$

$$
\begin{bmatrix}
0.650 & 0.000 & 0.000 & 0.025 & -0.350 & -0.325 \\
0.000 & 0.650 & 0.025 & 0.000 & -0.350 & -0.325 \\
0.000 & 0.025 & 0.650 & 0.000 & -0.325 & -0.350 \\
0.025 & 0.000 & 0.000 & 0.650 & -0.325 & -0.350 \\
-0.350 & -0.350 & -0.325 & -0.325 & 1.300 & 0.050 \\
-0.325 & -0.325 & -0.350 & -0.350 & 0.050 & 1.300
\end{bmatrix}
\begin{bmatrix} 20 \\ \Theta_2 \\ \Theta_3 \\ 20 \\ \Theta_5 \\ \Theta_6 \end{bmatrix} =
$$

$$
= \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 6 \\ 6 \end{bmatrix} + \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \\ 0 \\ 10 \end{bmatrix}
$$

Solution: $\Theta_2 = 48.429,\ \Theta_3 = 56.756,\ \Theta_5 = 40.361,\ \Theta_6 = 48.528,$

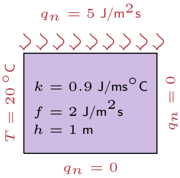
$f_{b1} = -19.398,\ f_{b4} = -24.602$

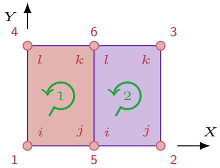
$$
\mathbf{K}^1 = \begin{bmatrix}
0.650 & -0.350 & -0.325 & 0.025 \\
-0.350 & 0.650 & 0.025 & -0.325 \\
-0.325 & 0.025 & 0.650 & -0.350 \\
0.025 & -0.325 & -0.350 & 0.650
\end{bmatrix}
$$

$$
\mathbf{K}^2 = \begin{bmatrix}
0.650 & -0.350 & -0.325 & 0.025 \\
-0.350 & 0.650 & 0.025 & -0.325 \\
-0.325 & 0.025 & 0.650 & -0.350 \\
0.025 & -0.325 & -0.350 & 0.650
\end{bmatrix}
$$

$$
\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}
$$

$$
\mathbf{f}_b^1 = \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 5 \end{bmatrix}
$$





2D example of heat flow – 4-node elements

Heat flux vector – element 1

$$\mathbf{\Theta}^1 = \begin{bmatrix} 20 \\ 40.361 \\ 48.528 \\ 20 \end{bmatrix}$$

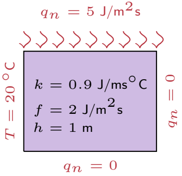
$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

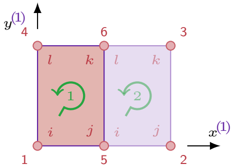
$$\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 5 \end{bmatrix}$$

$$\mathbf{\Theta} = \begin{bmatrix} 20 \\ 48.429 \\ 56.756 \\ 20 \\ 40.361 \\ 48.528 \end{bmatrix}$$





2D example of heat flow – 4-node elements

Heat flux vector – element 1

$$\boldsymbol{\Theta}^1 = \begin{bmatrix} 20 \\ 40.361 \\ 48.528 \\ 20 \end{bmatrix}$$

$$\mathbf{q}^1 = -k\mathbf{B}^1\boldsymbol{\Theta}^1$$

$$= -0.9 \begin{bmatrix} \frac{y^{(1)}-3}{6} & \frac{y^{(1)}-3}{-6} & \frac{y^{(1)}}{6} & \frac{y^{(1)}}{-6} \\ \frac{x^{(1)}-2}{6} & \frac{x^{(1)}}{-6} & \frac{x^{(1)}}{6} & \frac{x^{(1)}-2}{-6} \end{bmatrix} \begin{bmatrix} 20 \\ 40.361 \\ 48.528 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} -1.225y - 9.192 \\ -1.225x \end{bmatrix}$$

e.g. at centre of mass $\mathbf{q}^1(1, 1.5) = \begin{bmatrix} -11.000 \\ -1.225 \end{bmatrix}$

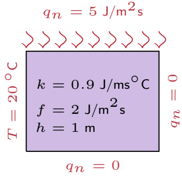
$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

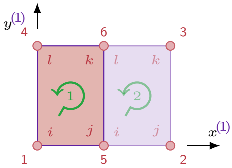
$$\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 5 \end{bmatrix}$$

$$\boldsymbol{\Theta} = \begin{bmatrix} 20 \\ 48.429 \\ 56.756 \\ 20 \\ 40.361 \\ 48.528 \end{bmatrix}$$

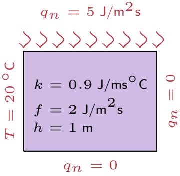


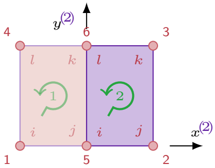


2D example of heat flow – 4-node elements

Heat flux vector – element 2

$$\boldsymbol{\Theta}^2 = \begin{bmatrix} 40.361 \\ 48.429 \\ 56.756 \\ 48.528 \end{bmatrix}$$





$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 5 \end{bmatrix}$$

$$\boldsymbol{\Theta} = \begin{bmatrix} 20 \\ 48.429 \\ 56.756 \\ 20 \\ 40.361 \\ 48.528 \end{bmatrix}$$

2D example of heat flow – 4-node elements

Heat flux vector – element 2

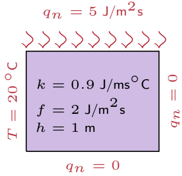
$$\boldsymbol{\Theta}^2 = \begin{bmatrix} 40.361 \\ 48.429 \\ 56.756 \\ 48.528 \end{bmatrix}$$

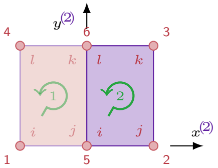
$$\mathbf{q}^2 = -k\mathbf{B}^2\boldsymbol{\Theta}^2$$

$$= -0.9 \begin{bmatrix} \frac{y^{(2)}-3}{6} & \frac{y^{(2)}-3}{-6} & \frac{y^{(2)}}{6} & \frac{y^{(2)}}{-6} \\ \frac{x^{(2)}-2}{6} & \frac{x^{(2)}}{-6} & \frac{x^{(2)}}{6} & \frac{x^{(2)}-2}{-6} \end{bmatrix} \begin{bmatrix} 40.361 \\ 48.429 \\ 56.756 \\ 48.528 \end{bmatrix}$$

$$= \begin{bmatrix} -0.024y - 3.631 \\ -0.024x - 2.450 \end{bmatrix}$$

e.g. at centre of mass $\mathbf{q}^2(1, 1.5) = \begin{bmatrix} -3.667 \\ -2.474 \end{bmatrix}$





$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 5 \end{bmatrix}$$

$$\boldsymbol{\Theta} = \begin{bmatrix} 20 \\ 48.429 \\ 56.756 \\ 20 \\ 40.361 \\ 48.528 \end{bmatrix}$$

2D example of heat flow – 4-node elements

Computation of temperature at a point in el. 1

$$\boldsymbol{\Theta}^1 = \begin{bmatrix} 20 \\ 40.361 \\ 48.528 \\ 20 \end{bmatrix}$$

$$T^e(x^e, y^e) = \mathbf{N}^e(x^e, y^e)\boldsymbol{\Theta}^e$$

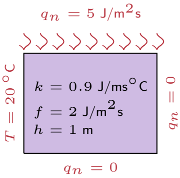
$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

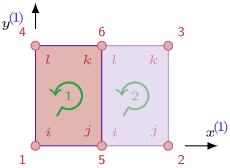
$$\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 5 \end{bmatrix}$$

$$\boldsymbol{\Theta} = \begin{bmatrix} 20 \\ 48.429 \\ 56.756 \\ 20 \\ 40.361 \\ 48.528 \end{bmatrix}$$





2D example of heat flow – 4-node elements

Computation of temperature at a point in el. 1

$$\boldsymbol{\Theta}^1 = \begin{bmatrix} 20 \\ 40.361 \\ 48.528 \\ 20 \end{bmatrix}$$

$$T^e(x^e, y^e) = \mathbf{N}^e(x^e, y^e)\boldsymbol{\Theta}^e$$

e.g. at centre of mass $(1, 1.5)$

$$T^1(1, 1.5) = \begin{bmatrix} \frac{(1-2)(1.5-3)}{6} & \frac{1(1.5-3)}{-6} & \frac{1 \cdot 1.5}{6} & \frac{(1-2)1.5}{-6} \end{bmatrix} \begin{bmatrix} 20 \\ 40.361 \\ 48.528 \\ 20 \end{bmatrix}$$

$$= 32.222$$

$$\mathbf{K}^1 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.650 & -0.350 & -0.325 & 0.025 \\ -0.350 & 0.650 & 0.025 & -0.325 \\ -0.325 & 0.025 & 0.650 & -0.350 \\ 0.025 & -0.325 & -0.350 & 0.650 \end{bmatrix}$$

$$\mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\mathbf{f}_b^1 = \begin{bmatrix} f_{b1} \\ 0 \\ 5 \\ 5 + f_{b4} \end{bmatrix}, \quad \mathbf{f}_b^2 = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 5 \end{bmatrix}$$

$$\boldsymbol{\Theta} = \begin{bmatrix} 20 \\ 48.429 \\ 56.756 \\ 20 \\ 40.361 \\ 48.528 \end{bmatrix}$$