Isoparametric elements

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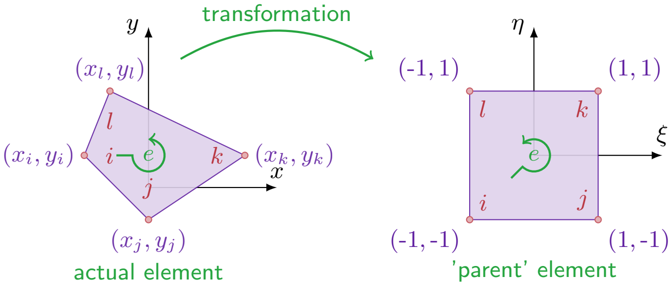
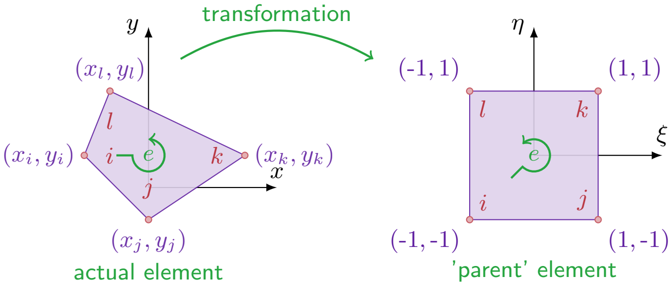
Isoparametric transformation



Isoparametric transformation



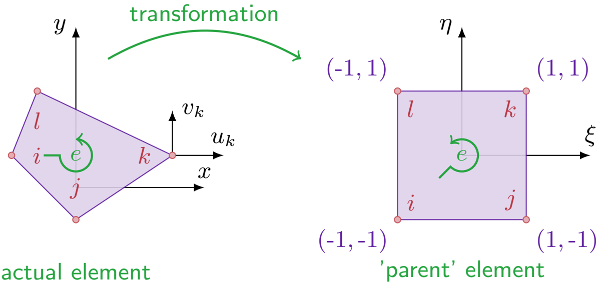
Element stiffness matrix and load vector

$$\mathbf{k}^e = \int_{A^e} \mathbf{B}^{e\mathrm{T}} \mathbf{D}^e \mathbf{B}^e h^e \mathrm{d}x\mathrm{d}y$$

$$\mathbf{p}^e = \int_{A^e} \mathbf{N}^{e\mathrm{T}} \mathbf{f}^e h^e \mathrm{d}x\mathrm{d}y$$

A^e, h^e – area and thickness of FE

Isoparametric transformation



Interpolation of displacement vector components

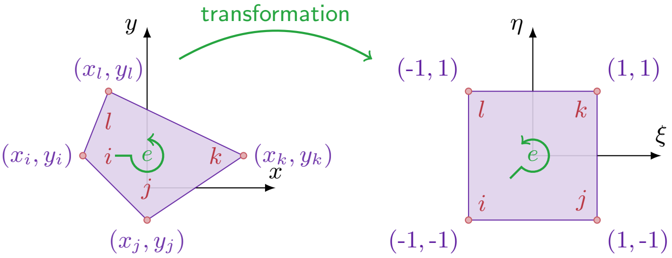
$$u(\xi,\eta) = \mathbf{N}(\xi,\eta)\mathbf{u}_n$$
$$v(\xi,\eta) = \mathbf{N}(\xi,\eta)\mathbf{v}_n$$

$$\mathbf{u}_n = \{u_i\ u_j\ u_k\ u_l\}$$
$$\mathbf{v}_n = \{v_i\ v_j\ v_k\ v_l\}$$

Shape functions

$$\mathbf{N}(\xi,\eta) = [N_i\ N_j\ N_k\ N_l]$$
$$N_i(\xi,\eta) = \tfrac{1}{4}(1-\xi)(1-\eta)$$
$$N_j(\xi,\eta) = \tfrac{1}{4}(1+\xi)(1-\eta)$$
$$N_k(\xi,\eta) = \tfrac{1}{4}(1+\xi)(1+\eta)$$
$$N_l(\xi,\eta) = \tfrac{1}{4}(1-\xi)(1+\eta)$$

Isoparametric transformation



actual element

'parent' element

Interpolation of actual (model) coordinates

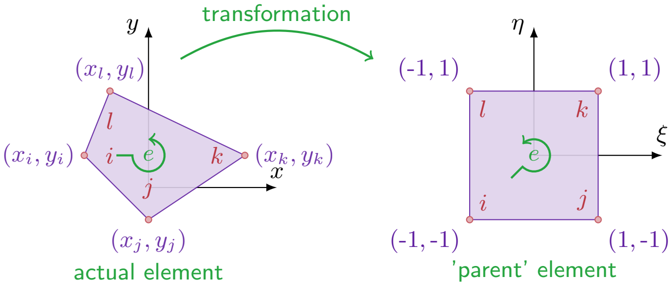
$$x(\xi, \eta) = \mathbf{N}(\xi, \eta)\mathbf{x}_n$$
$$y(\xi, \eta) = \mathbf{N}(\xi, \eta)\mathbf{y}_n$$

$$\mathbf{x}_n = \{x_i \; x_j \; x_k \; x_l\}$$
$$\mathbf{y}_n = \{y_i \; y_j \; y_k \; y_l\}$$

Shape functions

$$\mathbf{N}(\xi, \eta) = [N_i \; N_j \; N_k \; N_l]$$
$$N_i(\xi, \eta) = \tfrac{1}{4}(1 - \xi)(1 - \eta)$$
$$N_j(\xi, \eta) = \tfrac{1}{4}(1 + \xi)(1 - \eta)$$
$$N_k(\xi, \eta) = \tfrac{1}{4}(1 + \xi)(1 + \eta)$$
$$N_l(\xi, \eta) = \tfrac{1}{4}(1 - \xi)(1 + \eta)$$

Isoparametric transformation



Transformation (chain rule)

$$\mathrm{d}x = \frac{\partial x}{\partial \xi}\mathrm{d}\xi + \frac{\partial x}{\partial \eta}\mathrm{d}\eta$$
$$\mathrm{d}y = \frac{\partial y}{\partial \xi}\mathrm{d}\xi + \frac{\partial y}{\partial \eta}\mathrm{d}\eta$$
$$\Longrightarrow \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \mathrm{d}\xi \\ \mathrm{d}\eta \end{bmatrix}$$

$\mathbf{J}$ – Jacobi matrix

Differentiation of multivariable function

Partial derivatives of function $f(\xi, \eta)$

$$\begin{aligned}\frac{\partial f}{\partial \xi} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi} \\ \frac{\partial f}{\partial \eta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}\end{aligned} \implies \begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{\mathbf{J}^{\mathrm{T}}} \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

Differentiation of multivariable function

Partial derivatives of function $f(\xi, \eta)$

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$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \mathbf{J}^{-\mathrm{T}} \begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix}$$

Differentiation of multivariable function

Partial derivatives of function $f(\xi, \eta)$

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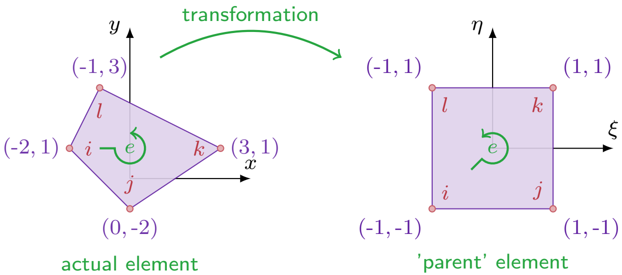
$$\begin{bmatrix}\frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y}\end{bmatrix} = \mathbf{J}^{-\mathrm{T}} \begin{bmatrix}\frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta}\end{bmatrix}$$

Integration

$$\int_A f(x, y)\,\mathrm{d}x\mathrm{d}y = \int_{-1}^{1}\int_{-1}^{1} f(x(\xi, \eta), y(\xi, \eta)) \det \mathbf{J}\,\mathrm{d}\xi\mathrm{d}\eta$$

Example - determination of conductivity matrix



Interpolation of actual (model) coordinates

$$x(\xi,\eta) = -2\cdot\frac{1}{4}(1-\xi)(1-\eta) + 0\cdot\frac{1}{4}(1+\xi)(1-\eta) + 3\cdot\frac{1}{4}(1+\xi)(1+\eta) - 1\cdot\frac{1}{4}(1-\xi)(1+\eta)$$

$$= \frac{3}{2}\xi + \eta + \frac{1}{2}\xi\eta$$

$$y(\xi,\eta) = 1\cdot\frac{1}{4}(1-\xi)(1-\eta) - 2\cdot\frac{1}{4}(1+\xi)(1-\eta) + 1\cdot\frac{1}{4}(1+\xi)(1+\eta) + 3\cdot\frac{1}{4}(1-\xi)(1+\eta)$$

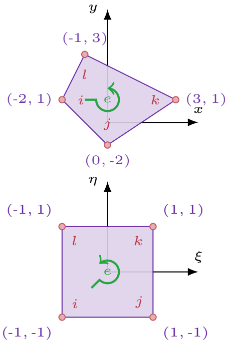
$$= \frac{3}{4} - \frac{5}{4}\xi + \frac{5}{4}\eta + \frac{1}{4}\xi\eta$$

Example - determination of conductivity matrix

Jacobian

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} + \frac{1}{2}\eta & 1 + \frac{1}{2}\xi \\ -\frac{5}{4} + \frac{1}{4}\eta & \frac{5}{4} + \frac{1}{4}\xi \end{bmatrix}$$

$$\det \mathbf{J} = \frac{25}{8} + \xi + \frac{3}{8}\eta$$



$$x(\xi,\eta) = \frac{3}{2}\xi + \eta + \frac{1}{2}\xi\eta$$

$$y(\xi,\eta) = \frac{3}{4} - \frac{5}{4}\xi + \frac{5}{4}\eta + \frac{1}{4}\xi\eta$$

Example - determination of conductivity matrix

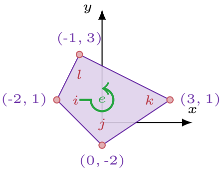
Jacobian

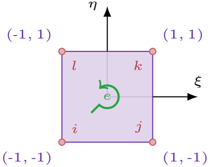
$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} + \frac{1}{2}\eta & 1 + \frac{1}{2}\xi \\ -\frac{5}{4} + \frac{1}{4}\eta & \frac{5}{4} + \frac{1}{4}\xi \end{bmatrix}$$

$$\det \mathbf{J} = \frac{25}{8} + \xi + \frac{3}{8}\eta$$

Conductivity matrix $k = 100, h = 0.1$

$$\mathbf{k} = \int_A \mathbf{B}^{\mathrm{T}} kh \mathbf{B} \mathrm{d}x \mathrm{d}y, \qquad \mathbf{B} = \begin{bmatrix} \dfrac{\partial \mathbf{N}}{\partial x} \\ \dfrac{\partial \mathbf{N}}{\partial y} \end{bmatrix} = \mathbf{J}^{-\mathrm{T}} \begin{bmatrix} \dfrac{\partial \mathbf{N}}{\partial \xi} \\ \dfrac{\partial \mathbf{N}}{\partial \eta} \end{bmatrix}$$





$$x(\xi,\eta) = \frac{3}{2}\xi + \eta + \frac{1}{2}\xi\eta$$

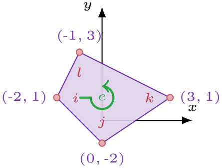
$$y(\xi,\eta) = \frac{3}{4} - \frac{5}{4}\xi + \frac{5}{4}\eta + \frac{1}{4}\xi\eta$$

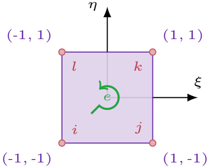
Example - determination of conductivity matrix

Shape function derivatives

$$\mathbf{J}^{-1} = \frac{1}{25 + 8\xi + 3\eta} \begin{bmatrix} 10 + 2\xi & -8 - 4\xi \\ 10 - 2\eta & 12 + 4\eta \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial \mathbf{N}}{\partial \xi} \\ \dfrac{\partial \mathbf{N}}{\partial \eta} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -1 + \eta & 1 - \eta & 1 + \eta & -1 - \eta \\ -1 + \xi & -1 - \xi & 1 + \xi & 1 - \xi \end{bmatrix}$$





$$x(\xi, \eta) = \frac{3}{2}\xi + \eta + \frac{1}{2}\xi\eta$$

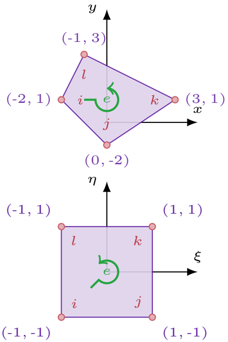
$$y(\xi, \eta) = \frac{3}{4} - \frac{5}{4}\xi + \frac{5}{4}\eta + \frac{1}{4}\xi\eta$$

Example - determination of conductivity matrix

Shape function derivatives

$$\mathbf{J}^{\text{-}1} = \frac{1}{25 + 8\xi + 3\eta} \begin{bmatrix} 10 + 2\xi & -8 - 4\xi \\ 10 - 2\eta & 12 + 4\eta \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial \mathbf{N}}{\partial \xi} \\ \dfrac{\partial \mathbf{N}}{\partial \eta} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -1 + \eta & 1 - \eta & 1 + \eta & -1 - \eta \\ -1 + \xi & -1 - \xi & 1 + \xi & 1 - \xi \end{bmatrix}$$



Matrix of shape function derivatives

$$\mathbf{B} = \frac{1}{25 + 8\xi + 3\eta} \begin{bmatrix} -5 + 2\xi + 3\eta & -2\xi - 2\eta & 5 + 3\xi + 2\eta & -3\xi - 3\eta \\ -1 + 4\xi - 3\eta & -5 - 4\xi + \eta & 1 + 2\xi - \eta & 5 - 2\xi + 3\eta \end{bmatrix}$$

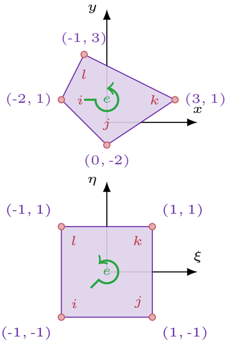
Example - determination of conductivity matrix

Conductivity matrix $k = 100, h = 0.1$

$$\mathbf{k} = \int_A \mathbf{B}(x,y)^{\mathrm{T}} kh \mathbf{B}(x,y) \mathrm{d}x\mathrm{d}y$$

$$= \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}(x(\xi,\eta), y(\xi,\eta))^{\mathrm{T}} kh \mathbf{B}(x(\xi,\eta), y(\xi,\eta)) \det \mathbf{J} \mathrm{d}\xi\mathrm{d}\eta$$

$$\mathbf{k} = \begin{bmatrix} 8.9488 & -1.0827 & -3.7421 & -4.1240 \\ -1.0827 & 6.1570 & -1.8099 & -3.2644 \\ -3.7421 & -1.8099 & 5.7670 & -0.2149 \\ -4.1240 & -3.2644 & -0.2149 & 7.6034 \end{bmatrix}$$



Matrix of shape function derivatives

$$\mathbf{B} = \frac{1}{25 + 8\xi + 3\eta} \begin{bmatrix} -5 + 2\xi + 3\eta & -2\xi - 2\eta & 5 + 3\xi + 2\eta & -3\xi - 3\eta \\ -1 + 4\xi - 3\eta & -5 - 4\xi + \eta & 1 + 2\xi - \eta & 5 - 2\xi + 3\eta \end{bmatrix}$$