Discretization error estimation

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Discretization error

Errors committed

- Modelling error
- Discretization error (of FEM approximation)
- Solution error

Methods of discretization error estimation

- hierarchical (Runge)
- explicit residual (implicit not considered here)
- based on averaging (Zienkiewicz-Zhu)
- interpolation error analysis (not considered here)

FE approximation using linear functions

Example problem

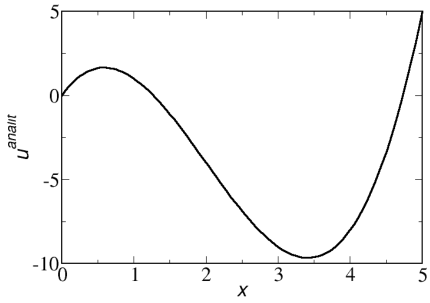
Solve BVP using 4 linear elements

$-u'' + u = f\,,\; f = x^3 - 6x^2 + 12\,,\; x \in (0,5)$

Bcs: $u(0) = 0\,,\;\; u(5) = 5$

Analytical solution:

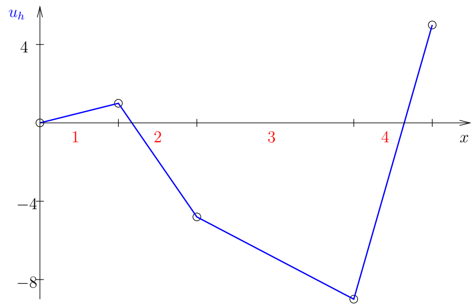
$u^{analit} = x^3 - 6x^2 + 6x$



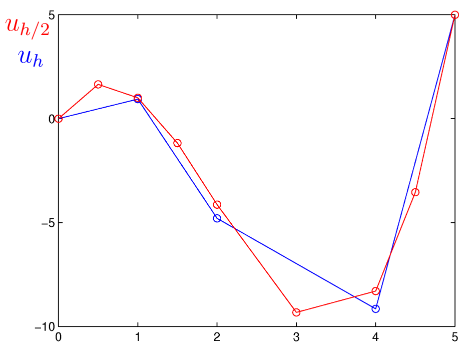
Approximate solution u_h (h - element size)

x	0	1	2	4	5
u_h	0	0.938	-4.797	9.153	5

Error measure: $e \stackrel{\text{def}}{=} u - u_h$



Hierarchical method $e^H = u_{h/2} - u_h$



x	0	0.5	1	1.5	2	3	4	4.5	5
u_h	0	0.469	0.938	-1.930	-4.797	-6.975	-9.153	-2.077	5
$u_{h/2}$	0	1.647	1.000	-1.179	-4.138	-9.324	-8.299	-3.543	5
$u_{h/2} - u_h$	0	1.178	0.062	0.751	0.660	-2.349	0.855	-1.467	0

$$\eta_i^H = \sqrt{\int_{x_i}^{x_{i+1}} (u_{h/2} - u_h)^2 \mathrm{d}x} \rightarrow \eta_1^H = 0.69,\; \eta_2^H = 0.59,\; \eta_3^H = 1.70,\; \eta_4^H = 0.79$$

$$||e^H||^2 = ||u_{h/2} - u_h||^2 = \int_0^5 (u_{h/2} - u_h)^2 \mathrm{d}x \rightarrow ||e^H|| \approx 2.08$$

$$||u_{h/2}||^2 = \int_0^5 (u_{h/2})^2 \mathrm{d}x \rightarrow ||u_{h/2}|| = 12.35 \rightarrow \frac{||e^H||}{||u_{h/2}||} \approx 17\%$$

Error estimation based on residuum (explicit)

Residuum of differential equation

$-u'' + u = f,\ f = x^3 - 6x^2 + 12 \rightarrow R(x) = f - (-u_h'' + u_h)$

Residuum provides bound on error

$||e|| \leqslant C||R||$

In 2D (J - jump of 1st derivative)

$||e||^2 \leqslant C(h^2||R||^2 + h||J||^2)$

Error indicator in 1D element i

$\eta_i^R = h_i\sqrt{\int_{x_i}^{x_{i+1}} R^2 \mathrm{d}x},\ \ u_h'' = 0 \rightarrow R = x^3 - 6x^2 + 12 - u_h$

Substitute interpolation for u_h, e.g.

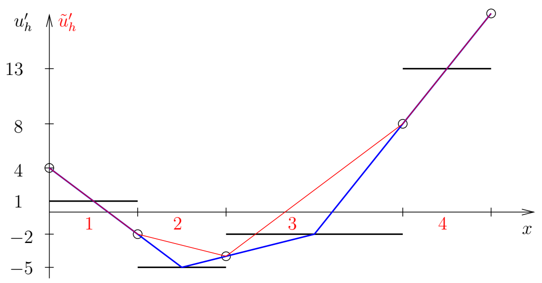
$\eta_1^R = 1\sqrt{\int_0^1 \{x^3 - 6x^2 + 12 - [0(x-1) + 0.938x]\}^2 \mathrm{d}x} = 9.94$

Compute relative error norm

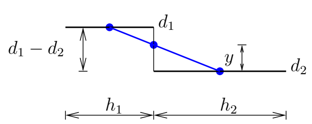
$\frac{\eta_1^R}{||f||} \approx 33\%$

and compare solution quality in elements

Error estimation based on smoothing $e^S = \tilde{u}_h' - u_h'$



Smoothed solution derivative $\tilde{u}_h'$ (through points determined at element edges)



$y = \frac{h_2}{h_1+h_2}(d_1 - d_2)$

$\tilde{u}' = d_2 + y = d_1\frac{h_2}{h_1+h_2} + d_2\frac{h_1}{h_1+h_2}$

If $h_1 = h_2$ then $\tilde{u}' = \frac{d_1+d_2}{2}$

$\tilde{u}_1' = 4.2$ (extrapolation to node 1),

$\tilde{u}_2' = -2.4,\ \tilde{u}_3' = -4.5,\ \tilde{u}_4' = 8.7,\ \tilde{u}_5' = 19.6$

$\eta_i^S = \sqrt{\int_{x_i}^{x_{i+1}} (\tilde{u}_h' - u_h')^2 \mathrm{d}x}$

$\eta_1^S = \sqrt{\int_0^1 \{[4.2(1-x) + (-2.4)x] - 0.94\}^2 \mathrm{d}x} = 1.93$

$\eta_2^S = 2.35,\ \eta_3^S = 8.09,\ \eta_4^S = 3.14$

$||e^S||^2 = \int_0^5 (\tilde{u}_h' - u_h')^2 \mathrm{d}x = \sum_i (\eta_i^S)^2 \rightarrow \frac{||e^S||}{||\tilde{u}_h'||} \approx 57\%$