FEM for buckling analysis

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Lecture contents

1 Introduction

2 Derivation of FEM equations
- Frame finite element

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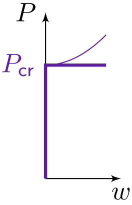
Introduction

Observation

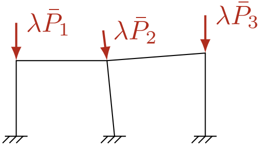
Bending stiffness is increased by tensile forces and decreased by compressive forces. A sufficiently large compressive force can reduce the bending stiffness to zero and structural buckling (instability mode) occurs.

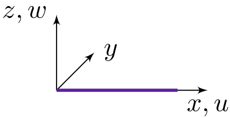


Assumptions

- linear elasticity: $\sigma_x = E\varepsilon_x$
- static one-parameter load
- ideal system (no imperfections)
- equilibrium of buckled configuration

$$\varepsilon_x(x) = u'(x) - zw''(x) + \frac{1}{2}\left(w'(x)\right)^2$$





Derivation of FEM equations

Energetic criterion of equilibrium

$$\Phi = U - W$$

- Φ – total energy
- U – elastic energy: $U = \frac{1}{2}\int\limits_V \varepsilon_x \sigma_x \mathrm{d}V$
- W – work of external forces: $W = \mathbf{d}^{\mathrm{T}}\mathbf{f}$
  - $\mathbf{d}$ – dof vector (nodal displacement vector)
  - $\mathbf{f}$ – external force vector

Derivation of FEM equations

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Value of (compressive) normal force before buckling

$$\frac{\Delta l}{l} = \varepsilon_x, \quad \sigma_x = \frac{N}{A}, \quad \sigma_x = E\varepsilon_x, \quad \varepsilon_x = u' \implies N(x) = EAu'(x)$$

Energetic criterion

Elastic energy

$$U = \frac{1}{2}\int_V \varepsilon_x \sigma_x \mathrm{d}V$$

Energetic criterion

Elastic energy

$$U = \frac{1}{2}\int_V \varepsilon_x \underset{E\varepsilon_x}{\sigma_x} \mathrm{d}V$$

Energetic criterion

Elastic energy

$$U = \frac{1}{2}\int_V E\varepsilon_x^2 \mathrm{d}V$$

Energetic criterion

Elastic energy

$$U = \frac{1}{2}\int_V E\varepsilon_x^2 \mathrm{d}V$$

$$U = \frac{1}{2}\int_V E\left(u' - zw'' + \frac{1}{2}w'^2\right)^2 \mathrm{d}V$$

Energetic criterion

Elastic energy

$$U = \frac{1}{2}\int_V E\varepsilon_x^2 \mathrm{d}V$$

$$U = \frac{1}{2}\int_V E\left(u' - zw'' + \frac{1}{2}w'^2\right)^2 \mathrm{d}V$$

Elastic energy of discretized system

$$U = \frac{1}{2}\sum_{e=1}^{E}\left\{\int_{V^e} E\left(u^{e\prime} - zw^{e\prime\prime} + \frac{1}{2}w^{e\prime 2}\right)^2 \mathrm{d}V^e\right\}$$

Energetic criterion

Elastic energy

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$$U=\frac{1}{2}\sum_{e=1}^{E}\left\{\int_0^{l^e}\left\{\int_{A^e} E\left(u^{e\prime 2}+z^2 w^{e\prime\prime 2}+\frac{1}{4}w^{e\prime 4}-2zu^{e\prime}w^{e\prime\prime}+u^{e\prime}w^{e\prime 2}-zw^{e\prime\prime}w^{e\prime 2}\right)\mathrm{d}A^e\right\}\mathrm{d}x^e\right\}$$

Energetic criterion

Elastic energy

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nonlinear term, upon linearization ≅ 0

Energetic criterion

Elastic energy

$$U = \frac{1}{2} \int_V E \varepsilon_x^2 \mathrm{d}V$$

$$U = \frac{1}{2} \int_V E \left(u' - z w'' + \frac{1}{2} w'^2 \right)^2 \mathrm{d}V$$

Elastic energy of discretized system

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$$U = \frac{1}{2} \sum_{e=1}^{E} \left\{ \int_0^{l^e} \left\{ \int_{A^e} E \left(u^{e\prime 2} + z^2 w^{e\prime\prime 2} - 2 z u^{e\prime} w^{e\prime\prime} + u^{e\prime} w^{e\prime 2} - z w^{e\prime\prime} w^{e\prime 2} \right) \mathrm{d}A^e \right\} \mathrm{d}x^e \right\}$$

Energetic criterion

Elastic energy

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$$U=\frac{1}{2}\sum_{e=1}^{E}\left\{\int_0^{l^e}\left(EA^e u^{e\prime 2}+EI_y^e w^{e\prime\prime 2}+EA^e u^{e\prime} w^{e\prime 2}\right)\mathrm{d}x^e\right\}$$

Energetic criterion

Elastic energy

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$$U=\frac{1}{2}\sum_{e=1}^{E}\left\{\int_0^{l^e}\left(EA^e u^{e\prime 2}+EI_y^e w^{e\prime\prime 2}+\underbrace{EA^e u^{e\prime}}_{\text{normal force } N^e(x)} w^{e\prime 2}\right)\mathrm{d}x^e\right\}$$

Energetic criterion

Elastic energy

$$U = \frac{1}{2}\int_V E\varepsilon_x^2 \mathrm{d}V$$

$$U = \frac{1}{2}\int_V E\left(u' - zw'' + \frac{1}{2}w'^2\right)^2 \mathrm{d}V$$

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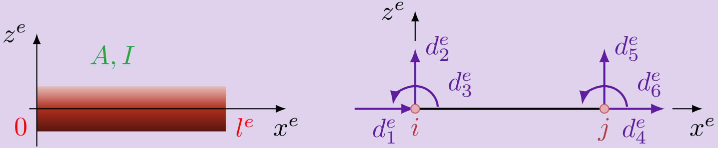
$$U=\frac{1}{2}\sum_{e=1}^{E}\left\{\int_0^{l^e}\left(EA^e u^{e\prime 2}+EI_y^e w^{e\prime\prime 2}+ N^e(x)w^{e\prime 2}\right)\mathrm{d}x^e\right\}$$

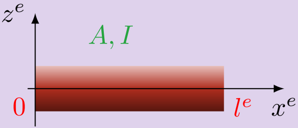
Discretization

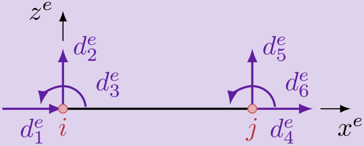
Frame finite element



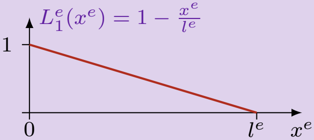
Discretization

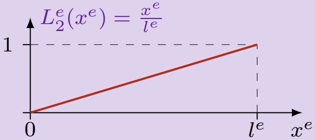
Frame finite element





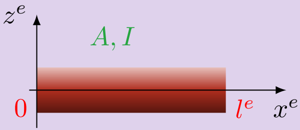
Displacement $u(x)^e = \mathbf{N}_u \mathbf{d}^e$, $\mathbf{N}_u = [L_1^e \;\; 0 \;\; 0 \;\; L_2^e \;\; 0 \;\; 0]$

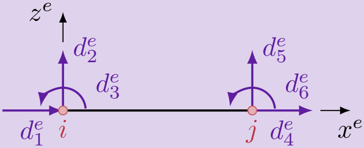




Discretization

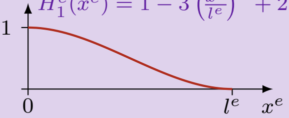
Frame finite element



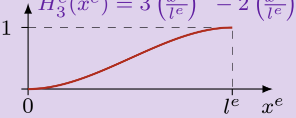


Deflection $w(x)^e = \mathbf{N}_w \mathbf{d}^e, \qquad \mathbf{N}_w = [0 \;\; H_1^e \;\; H_2^e \;\; 0 \;\; H_3^e \;\; H_4^e]$

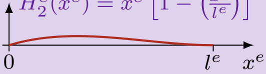
$$H_1^e(x^e) = 1 - 3\left(\frac{x^e}{l^e}\right)^2 + 2\left(\frac{x^e}{l^e}\right)^3$$



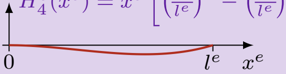
$$H_3^e(x^e) = 3\left(\frac{x^e}{l^e}\right)^2 - 2\left(\frac{x^e}{l^e}\right)^3$$



$$H_2^e(x^e) = x^e\left[1 - \left(\frac{x^e}{l^e}\right)\right]^2$$



$$H_4^e(x^e) = x^e\left[\left(\frac{x^e}{l^e}\right)^2 - \left(\frac{x^e}{l^e}\right)\right]$$



Approximation

Elastic energy for discretized structure

$$U = \frac{1}{2}\sum_{e=1}^{E}\left\{\int_0^{l^e}\left(EA^e u^{e\prime 2} + EI_y^e w^{e\prime\prime 2} + N^e(x)w^{e\prime 2}\right)\mathrm{d}x^e\right\}$$

$$u(x)^e = \mathbf{N}_u\mathbf{d}^e = \mathbf{N}_u\mathbf{T}^e\mathbf{d}, \qquad w(x)^e = \mathbf{N}_w\mathbf{d}^e = \mathbf{N}_w\mathbf{T}^e\mathbf{d}$$

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$$\begin{aligned} U = \frac{1}{2}\sum_{e=1}^{E}\Bigg\{\int_0^{l^e}\Big(& EA^e \mathbf{d}^{\mathrm{T}} \mathbf{T}^{e\,\mathrm{T}} \mathbf{N}_u'^{\,\mathrm{T}} \mathbf{N}_u' \mathbf{T}^e \mathbf{d} + EI_y^e \mathbf{d}^{\mathrm{T}} \mathbf{T}^{e\,\mathrm{T}} \mathbf{N}_w''^{\,\mathrm{T}} \mathbf{N}_w'' \mathbf{T}^e \mathbf{d} \\ & + N^e(x) \mathbf{d}^{\mathrm{T}} \mathbf{T}^{e\,\mathrm{T}} \mathbf{N}_w'^{\,\mathrm{T}} \mathbf{N}_w' \mathbf{T}^e \mathbf{d}\Big)\mathrm{d}x^e\Bigg\} \end{aligned}$$

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$$U = \frac{1}{2}\mathbf{d}^{\mathrm{T}}\sum_{e=1}^{E}\left\{\int_0^{l^e}\left(EA^e \mathbf{T}^{e\,\mathrm{T}} \mathbf{N}_u'^{\,\mathrm{T}} \mathbf{N}_u' \mathbf{T}^e + EI_y^e \mathbf{T}^{e\,\mathrm{T}} \mathbf{N}_w''^{\,\mathrm{T}} \mathbf{N}_w'' \mathbf{T}^e \right.\right.$$

$$\left.\left. + \; N^e(x) \mathbf{T}^{e\,\mathrm{T}} \mathbf{N}_w'^{\,\mathrm{T}} \mathbf{N}_w' \mathbf{T}^e\right)\mathrm{d}x^e\right\}\mathbf{d}$$

Approximation

Elastic energy for discretized structure

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$$u(x)^e = \mathbf{N}_u\mathbf{d}^e = \mathbf{N}_u\mathbb{T}^e\mathbf{d}, \qquad w(x)^e = \mathbf{N}_w\mathbf{d}^e = \mathbf{N}_w\mathbb{T}^e\mathbf{d}$$

$$\begin{aligned} U = \frac{1}{2}\mathbf{d}^{\mathrm{T}}\Bigg\{\sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\int_0^{l^e}\left(EA^e\mathbf{N}_u'^{\mathrm{T}}\mathbf{N}_u' + EI_y^e\mathbf{N}_w''^{\mathrm{T}}\mathbf{N}_w''\right)\mathrm{d}x^e\mathbb{T}^e \\ + \sum_{e=1}^{E}\mathbb{T}^{e\mathrm{T}}\int_0^{l^e}N^e(x)\mathbf{N}_w'^{\mathrm{T}}\mathbf{N}_w'\mathrm{d}x^e\mathbb{T}^e\Bigg\}\mathbf{d} \end{aligned}$$

Approximation

Elastic energy for discretized structure

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$$U=\frac{1}{2}\mathbf{d}^{\mathrm{T}}\left\{\sum_{e=1}^{E}\mathbb{T}^{e\,\mathrm{T}}\overbrace{\int_0^{l^e}\left(EA^e\mathbf{N}_u'^{\,\mathrm{T}}\mathbf{N}_u'+EI_y^e\mathbf{N}_w''^{\,\mathrm{T}}\mathbf{N}_w''\right)\mathrm{d}x^e}^{\mathbf{K}^e\text{ – linear stiffness matrix}}\;\mathbb{T}^e\right.$$

$$\left.+\sum_{e=1}^{E}\mathbb{T}^{e\,\mathrm{T}}\underbrace{\int_0^{l^e}N^e(x)\mathbf{N}_w'^{\,\mathrm{T}}\mathbf{N}_w'\mathrm{d}x^e}_{\mathbf{K}_\sigma^e\text{ – initial stress matrix}}\;\mathbb{T}^e\right\}\mathbf{d}$$

Approximation

Elastic energy for discretized structure

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$$U=\frac{1}{2}\mathbf{d}^{\mathrm{T}}\left(\sum_{e=1}^{E}\mathbb{T}^{e\,\mathrm{T}}\mathbf{K}^e\mathbb{T}^e+\sum_{e=1}^{E}\mathbb{T}^{e\,\mathrm{T}}\mathbf{K}_\sigma^e\mathbb{T}^e\right)\mathbf{d}$$

Approximation

Elastic energy for discretized structure

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$$U = \frac{1}{2}\mathbf{d}^{\mathrm{T}}\left(\underbrace{\sum_{e=1}^{E}\mathbb{T}^{e\,\mathrm{T}}\mathbf{K}^e\mathbb{T}^e}_{\mathbf{K}} + \underbrace{\sum_{e=1}^{E}\mathbb{T}^{e\,\mathrm{T}}\mathbf{K}_\sigma^e\mathbb{T}^e}_{\mathbf{K}_\sigma}\right)\mathbf{d}$$

Approximation

Elastic energy for discretized structure

$$U = \frac{1}{2}\sum_{e=1}^{E}\left\{\int_0^{l^e}\left(EA^e u^{e\prime 2} + EI_y^e w^{e\prime\prime 2} + N^e(x) w^{e\prime 2}\right)\mathrm{d}x^e\right\}$$

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$$U = \frac{1}{2}\mathbf{d}^{\mathrm{T}}\left(\mathbf{K} + \mathbf{K}_\sigma\right)\mathbf{d}$$

FEM equations for stability

Total energy

$$\Phi = \frac{1}{2}\mathbf{d}^{\mathrm{T}}\left(\mathbf{K} + \mathbf{K}_{\sigma}\right)\mathbf{d} - \mathbf{d}^{\mathrm{T}}\mathbf{f}$$

FEM equations for stability

Total energy

$$\Phi = \frac{1}{2}\mathbf{d}^{\mathrm{T}}\left(\mathbf{K} + \mathbf{K}_{\sigma}\right)\mathbf{d} - \mathbf{d}^{\mathrm{T}}\mathbf{f}$$

Minimization of energy

$$\delta\Phi = 0 \Longrightarrow \left(\mathbf{K} + \mathbf{K}_{\sigma}\right)\mathbf{d} - \mathbf{f} = 0$$

FEM equations for stability

Total energy

$$\Phi = \frac{1}{2}\mathbf{d}^{\mathrm{T}}\left(\mathbf{K}+\mathbf{K}_{\sigma}\right)\mathbf{d} - \mathbf{d}^{\mathrm{T}}\mathbf{f}$$

Minimization of energy

$$\delta\Phi = 0 \Longrightarrow \left(\mathbf{K}+\mathbf{K}_{\sigma}\right)\mathbf{d} - \mathbf{f} = 0$$

Equations for two adjacent equilibrium states – before and after buckling – eigenproblem

$$\left.\begin{array}{l} (\mathbf{K}+\mathbf{K}_{\sigma})\mathbf{d}^{1} = \mathbf{f} \\ (\mathbf{K}+\mathbf{K}_{\sigma})\mathbf{d}^{2} = \mathbf{f} \end{array}\right/ - \quad \Longrightarrow (\mathbf{K}+\mathbf{K}_{\sigma})\Delta\mathbf{d} = 0$$

FEM equations for stability

Total energy

$$\Phi = \frac{1}{2}\mathbf{d}^{\mathrm{T}}\left(\mathbf{K}+\mathbf{K}_{\sigma}\right)\mathbf{d}-\mathbf{d}^{\mathrm{T}}\mathbf{f}$$

Minimization of energy

$$\delta\Phi = 0 \Longrightarrow \left(\mathbf{K}+\mathbf{K}_{\sigma}\right)\mathbf{d}-\mathbf{f}=0$$

Equations for two adjacent equilibrium states – before and after buckling – eigenproblem

$$\left.\begin{array}{l}(\mathbf{K}+\mathbf{K}_{\sigma})\mathbf{d}^{1}=\mathbf{f}\\ (\mathbf{K}+\mathbf{K}_{\sigma})\mathbf{d}^{2}=\mathbf{f}\end{array}\right/ - \qquad \Longrightarrow (\mathbf{K}+\mathbf{K}_{\sigma})\Delta\mathbf{d}=0$$

equation is satisfied when

$$\det\left(\mathbf{K}+\mathbf{K}_{\sigma}\right)=0 \text{ or } \Delta\mathbf{d}=0$$

FEM equations for stability

Total energy

$$\Phi = \frac{1}{2}\mathbf{d}^{\mathrm{T}}\left(\mathbf{K} + \mathbf{K}_\sigma\right)\mathbf{d} - \mathbf{d}^{\mathrm{T}}\mathbf{f}$$

Minimization of energy

$$\delta\Phi = 0 \Longrightarrow \left(\mathbf{K} + \mathbf{K}_\sigma\right)\mathbf{d} - \mathbf{f} = 0$$

Equations for two adjacent equilibrium states – before and after buckling – eigenproblem

$$\begin{array}{l} (\mathbf{K} + \mathbf{K}_\sigma)\mathbf{d}^1 = \mathbf{f} \\ (\mathbf{K} + \mathbf{K}_\sigma)\mathbf{d}^2 = \mathbf{f} \end{array} \Big/ - \quad \Longrightarrow (\mathbf{K} + \mathbf{K}_\sigma)\Delta\mathbf{d} = 0$$

equation is satisfied when

$$\det\left(\mathbf{K} + \mathbf{K}_\sigma\right) = 0$$

FEM for frame buckling

Linear stiffness matrix – frame element

$$\mathbf{K}^e = \int_0^{l^e} \mathbf{B}^{e\mathrm{T}} \mathbf{D}^e \mathbf{B}^e \mathrm{d}x^e$$

$$\mathbf{N}^e = \begin{bmatrix} \mathbf{N}_u^e \\ \mathbf{N}_w^e \end{bmatrix}, \quad \mathbf{B}^e = \mathbf{L}\mathbf{N}^e, \quad \mathbf{L} = \begin{bmatrix} \dfrac{\mathrm{d}}{\mathrm{d}x^e} \\ -\dfrac{\mathrm{d}^2}{\mathrm{d}x^{e2}} \end{bmatrix}, \quad \mathbf{D}^e = \begin{bmatrix} EA^e & 0 \\ 0 & EI^e \end{bmatrix}$$

$$\mathbf{K}^e = \begin{bmatrix} \dfrac{EA}{l} & 0 & 0 & -\dfrac{EA}{l} & 0 & 0 \\ 0 & \dfrac{12EI}{l^3} & \dfrac{6EI}{l^2} & 0 & -\dfrac{12EI}{l^3} & \dfrac{6EI}{l^2} \\ 0 & \dfrac{6EI}{l^2} & \dfrac{4EI}{l} & 0 & -\dfrac{6EI}{l^2} & \dfrac{2EI}{l} \\ -\dfrac{EA}{l} & 0 & 0 & \dfrac{EA}{l} & 0 & 0 \\ 0 & -\dfrac{12EI}{l^3} & -\dfrac{6EI}{l^2} & 0 & \dfrac{12EI}{l^3} & -\dfrac{6EI}{l^2} \\ 0 & \dfrac{6EI}{l^2} & \dfrac{2EI}{l} & 0 & -\dfrac{6EI}{l^2} & \dfrac{4EI}{l} \end{bmatrix}^e$$

FEM for frame buckling

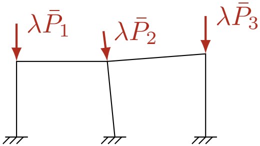
Initial stress matrix – frame element

$$\mathbf{K}_{\sigma}^{e} = \int_{0}^{l^{e}} N^{e}(x)\mathbf{N}_{w}'^{\mathrm{T}}\mathbf{N}_{w}'\mathrm{d}x^{e}$$

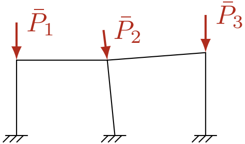
$$\mathbf{k}_{\sigma}^{e} = \frac{N^{e}(x)}{30l^{e}} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 3l & 0 & -36 & 3l \\ 0 & 3l & 4l^2 & 0 & -3l & -l^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -36 & -3l & 0 & 36 & -3l \\ 0 & 3l & -l^2 & 0 & -3l & 4l^2 \end{bmatrix}^{e}$$

$$\mathbf{f} = \lambda\bar{\mathbf{f}} \Longrightarrow \mathbf{k}_{\sigma}^{e} = \lambda\bar{\mathbf{k}}_{\sigma}^{e}$$

One-parameter loading $\mathbf{f}$



Configurational loading $\bar{\mathbf{f}}$



Frame buckling

FEM algorithm

1. Statics – determination of normal forces

$$\mathbf{K}\mathbf{d} = \bar{\mathbf{f}} \Longrightarrow N^e \Longrightarrow \bar{\mathbf{k}}_{\sigma}^{e}$$

2. Buckling – eigenproblem

$$(\mathbf{K} + \lambda \bar{\mathbf{K}}_{\sigma})\Delta\mathbf{d} = \mathbf{0} \Longrightarrow \lambda_{\mathsf{kr}} \Longrightarrow \Delta\mathbf{d} \text{ – buckling mode}$$

Example

Cantilever

Statics

After computations of pre-buckling state:

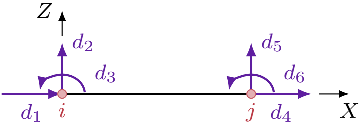
$$N(x) = -1$$

Example

Cantilever

Buckling

$$(\mathbf{K} + \lambda \bar{\mathbf{K}}_\sigma)\Delta \mathbf{d} = \mathbf{0}$$



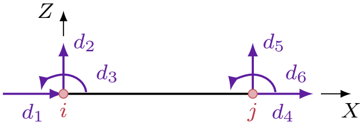
Example

Cantilever

Buckling

$$(\mathbf{K}+\lambda\bar{\mathbf{K}}_\sigma)\Delta\mathbf{d}=\mathbf{0}$$

$$\left(\frac{EI}{l^3}\begin{bmatrix} \frac{Al^2}{I} & 0 & 0 & -\frac{Al^2}{I} & 0 & 0 \\ 0 & 12 & 6l & 0 & -12 & 6l \\ 0 & 6l & 4l^2 & 0 & -6l & 2l^2 \\ -\frac{Al^2}{I} & 0 & 0 & \frac{Al^2}{I} & 0 & 0 \\ 0 & -12 & -6l & 0 & 12 & -6l \\ 0 & 6l & 2l^2 & 0 & -6l & 4l^2 \end{bmatrix} + \lambda\frac{1}{30l}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 3l & 0 & -36 & 3l \\ 0 & 3l & 4l^2 & 0 & -3l & -l^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -36 & -3l & 0 & 36 & -3l \\ 0 & 3l & -l^2 & 0 & -3l & 4l^2 \end{bmatrix}\right)\begin{bmatrix} \Delta\mathbf{d}_1 \\ \Delta\mathbf{d}_2 \\ \Delta\mathbf{d}_3 \\ \Delta\mathbf{d}_4 \\ \Delta\mathbf{d}_5 \\ \Delta\mathbf{d}_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



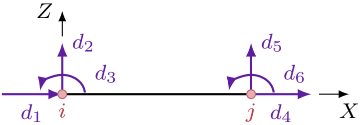
Example

Cantilever

Buckling

$$(\mathbf{K}+\lambda\bar{\mathbf{K}}_{\sigma})\Delta\mathbf{d}=\mathbf{0}$$

$$\left(\frac{EI}{l^3}\begin{bmatrix} \frac{Al^2}{I} & 0 & 0 & -\frac{Al^2}{I} & 0 & 0 \\ 0 & 12 & 6l & 0 & -12 & 6l \\ 0 & 6l & 4l^2 & 0 & -6l & 2l^2 \\ -\frac{Al^2}{I} & 0 & 0 & \frac{Al^2}{I} & 0 & 0 \\ 0 & -12 & -6l & 0 & 12 & -6l \\ 0 & 6l & 2l^2 & 0 & -6l & 4l^2 \end{bmatrix} + \lambda\frac{1}{30l}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 3l & 0 & -36 & 3l \\ 0 & 3l & 4l^2 & 0 & -3l & -l^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -36 & -3l & 0 & 36 & -3l \\ 0 & 3l & -l^2 & 0 & -3l & 4l^2 \end{bmatrix}\right)\begin{bmatrix} {\color{red}0} \\ {\color{red}0} \\ {\color{red}0} \\ \Delta\mathbf{d}_4 \\ \Delta\mathbf{d}_5 \\ \Delta\mathbf{d}_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



Example

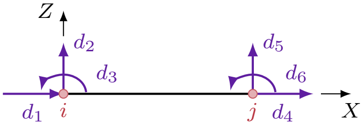
Cantilever

Buckling

$$(\mathbf{K} + \lambda \bar{\mathbf{K}}_\sigma)\Delta\mathbf{d} = \mathbf{0}$$

$$\left(\frac{EI}{l^3}\begin{bmatrix} \frac{Al^2}{I} & 0 & 0 & -\frac{Al^2}{I} & 0 & 0 \\ 0 & 12 & 6l & 0 & -12 & 6l \\ 0 & 6l & 4l^2 & 0 & -6l & 2l^2 \\ -\frac{Al^2}{I} & 0 & 0 & \frac{Al^2}{I} & 0 & 0 \\ 0 & -12 & -6l & 0 & 12 & -6l \\ 0 & 6l & 2l^2 & 0 & -6l & 4l^2 \end{bmatrix} + \lambda \frac{1}{30l}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 3l & 0 & -36 & 3l \\ 0 & 3l & 4l^2 & 0 & -3l & -l^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -36 & -3l & 0 & 36 & -3l \\ 0 & 3l & -l^2 & 0 & -3l & 4l^2 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 0 \\ 0 \\ \Delta\mathbf{d}_4 \\ \Delta\mathbf{d}_5 \\ \Delta\mathbf{d}_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\frac{EI}{l^3}\begin{bmatrix} \frac{Al^2}{I} & 0 & 0 \\ 0 & 12 & -6l \\ 0 & -6l & 4l^2 \end{bmatrix} + \lambda \frac{1}{30l}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 36 & -3l \\ 0 & -3l & 4l^2 \end{bmatrix}\right)\begin{bmatrix} \Delta\mathbf{d}_4 \\ \Delta\mathbf{d}_5 \\ \Delta\mathbf{d}_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



Example

Cantilever

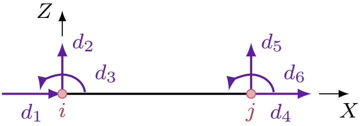
Buckling

$$(\mathbf{K}+\lambda\bar{\mathbf{K}}_{\sigma})\Delta\mathbf{d}=\mathbf{0}$$

$$\left(\frac{EI}{l^3}\begin{bmatrix} \frac{Al^2}{I} & 0 & 0 & -\frac{Al^2}{I} & 0 & 0 \\ 0 & 12 & 6l & 0 & -12 & 6l \\ 0 & 6l & 4l^2 & 0 & -6l & 2l^2 \\ -\frac{Al^2}{I} & 0 & 0 & \frac{Al^2}{I} & 0 & 0 \\ 0 & -12 & -6l & 0 & 12 & -6l \\ 0 & 6l & 2l^2 & 0 & -6l & 4l^2 \end{bmatrix}+\lambda\frac{1}{30l}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 3l & 0 & -36 & 3l \\ 0 & 3l & 4l^2 & 0 & -3l & -l^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -36 & -3l & 0 & 36 & -3l \\ 0 & 3l & -l^2 & 0 & -3l & 4l^2 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 0 \\ 0 \\ \Delta\mathbf{d}_4 \\ \Delta\mathbf{d}_5 \\ \Delta\mathbf{d}_6 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\frac{EI}{l^3}\begin{bmatrix} \frac{Al^2}{I} & 0 & 0 \\ 0 & 12 & -6l \\ 0 & -6l & 4l^2 \end{bmatrix}+\lambda\frac{1}{30l}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 36 & -3l \\ 0 & -3l & 4l^2 \end{bmatrix}\right)\begin{bmatrix} \Delta\mathbf{d}_4 \\ \Delta\mathbf{d}_5 \\ \Delta\mathbf{d}_6 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left|\frac{EI}{l^3}\begin{bmatrix} \frac{Al^2}{I} & 0 & 0 \\ 0 & 12 & -6l \\ 0 & -6l & 4l^2 \end{bmatrix}+\lambda\frac{1}{30l}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 36 & -3l \\ 0 & -3l & 4l^2 \end{bmatrix}\right|=0$$



Example

Cantilever

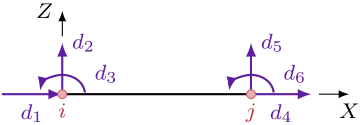
Buckling

$$(\mathbf{K} + \lambda \bar{\mathbf{K}}_\sigma)\Delta\mathbf{d} = \mathbf{0}$$

$$\left(\frac{EI}{l^3}\begin{bmatrix} \frac{Al^2}{I} & 0 & 0 & -\frac{Al^2}{I} & 0 & 0 \\ 0 & 12 & 6l & 0 & -12 & 6l \\ 0 & 6l & 4l^2 & 0 & -6l & 2l^2 \\ -\frac{Al^2}{I} & 0 & 0 & \frac{Al^2}{I} & 0 & 0 \\ 0 & -12 & -6l & 0 & 12 & -6l \\ 0 & 6l & 2l^2 & 0 & -6l & 4l^2 \end{bmatrix} + \lambda\frac{1}{30l}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 3l & 0 & -36 & 3l \\ 0 & 3l & 4l^2 & 0 & -3l & -l^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -36 & -3l & 0 & 36 & -3l \\ 0 & 3l & -l^2 & 0 & -3l & 4l^2 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 0 \\ 0 \\ \Delta\mathbf{d}_4 \\ \Delta\mathbf{d}_5 \\ \Delta\mathbf{d}_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\frac{EI}{l^3}\begin{bmatrix} \frac{Al^2}{I} & 0 & 0 \\ 0 & 12 & -6l \\ 0 & -6l & 4l^2 \end{bmatrix} + \lambda\frac{1}{30l}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 36 & -3l \\ 0 & -3l & 4l^2 \end{bmatrix}\right)\begin{bmatrix} \Delta\mathbf{d}_4 \\ \Delta\mathbf{d}_5 \\ \Delta\mathbf{d}_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left|\frac{EI}{l^3}\begin{bmatrix} \frac{Al^2}{I} & 0 & 0 \\ 0 & 12 & -6l \\ 0 & -6l & 4l^2 \end{bmatrix} + \lambda\frac{1}{30l}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 36 & -3l \\ 0 & -3l & 4l^2 \end{bmatrix}\right| = 0 \Rightarrow \begin{aligned} \lambda_1 &= 2.486\frac{EI}{l^2} \\ \lambda_2 &= 32.181\frac{EI}{l^2} \end{aligned}$$



Example

Cantilever

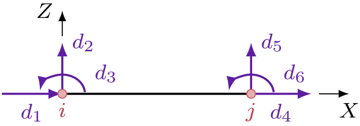
Buckling

$$(\mathbf{K}+\lambda\bar{\mathbf{K}}_{\sigma})\Delta\mathbf{d}=\mathbf{0}$$

$$\left(\frac{EI}{l^3}\begin{bmatrix} \frac{Al^2}{I} & 0 & 0 & -\frac{Al^2}{I} & 0 & 0 \\ 0 & 12 & 6l & 0 & -12 & 6l \\ 0 & 6l & 4l^2 & 0 & -6l & 2l^2 \\ -\frac{Al^2}{I} & 0 & 0 & \frac{Al^2}{I} & 0 & 0 \\ 0 & -12 & -6l & 0 & 12 & -6l \\ 0 & 6l & 2l^2 & 0 & -6l & 4l^2 \end{bmatrix}+\lambda\frac{1}{30l}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 3l & 0 & -36 & 3l \\ 0 & 3l & 4l^2 & 0 & -3l & -l^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -36 & -3l & 0 & 36 & -3l \\ 0 & 3l & -l^2 & 0 & -3l & 4l^2 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 0 \\ 0 \\ \Delta\mathbf{d}_4 \\ \Delta\mathbf{d}_5 \\ \Delta\mathbf{d}_6 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\frac{EI}{l^3}\begin{bmatrix} \frac{Al^2}{I} & 0 & 0 \\ 0 & 12 & -6l \\ 0 & -6l & 4l^2 \end{bmatrix}+\lambda\frac{1}{30l}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 36 & -3l \\ 0 & -3l & 4l^2 \end{bmatrix}\right)\begin{bmatrix} \Delta\mathbf{d}_4 \\ \Delta\mathbf{d}_5 \\ \Delta\mathbf{d}_6 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left|\frac{EI}{l^3}\begin{bmatrix} \frac{Al^2}{I} & 0 & 0 \\ 0 & 12 & -6l \\ 0 & -6l & 4l^2 \end{bmatrix}+\lambda\frac{1}{30l}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 36 & -3l \\ 0 & -3l & 4l^2 \end{bmatrix}\right|=0\Rightarrow \begin{array}{l} \lambda_1=2.486\dfrac{EI}{l^2}\Rightarrow P_{\text{cr}}^{\text{anal}}=2.467\dfrac{EI}{l^2} \\ \lambda_2=32.181\dfrac{EI}{l^2} \end{array}$$



Example

Cantilever

Buckling modes

Buckling modes are determined from one of two linearly dependent equations upon substitution of respective eigenvalue