Introduction to Computational Methods

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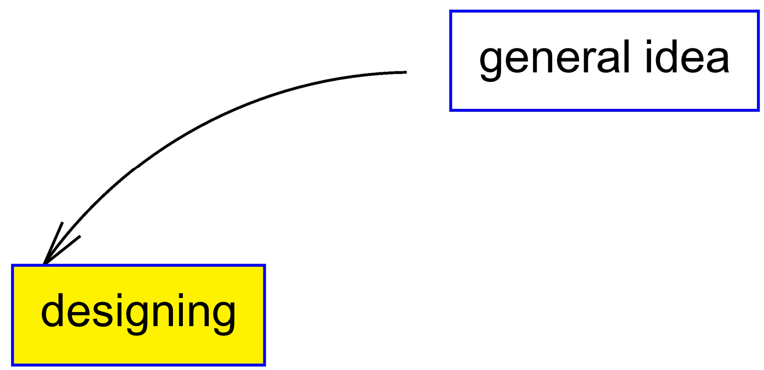
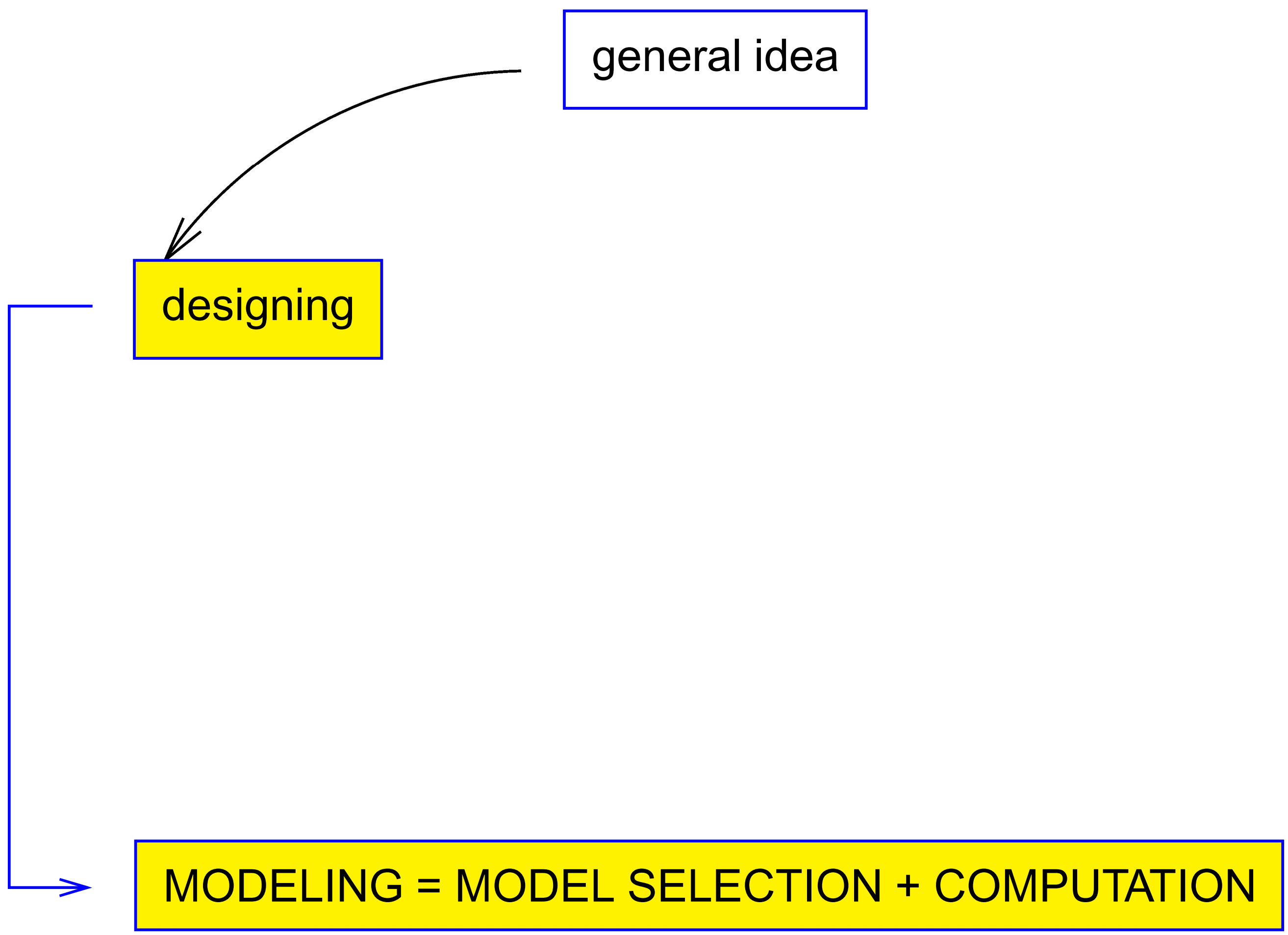
general idea

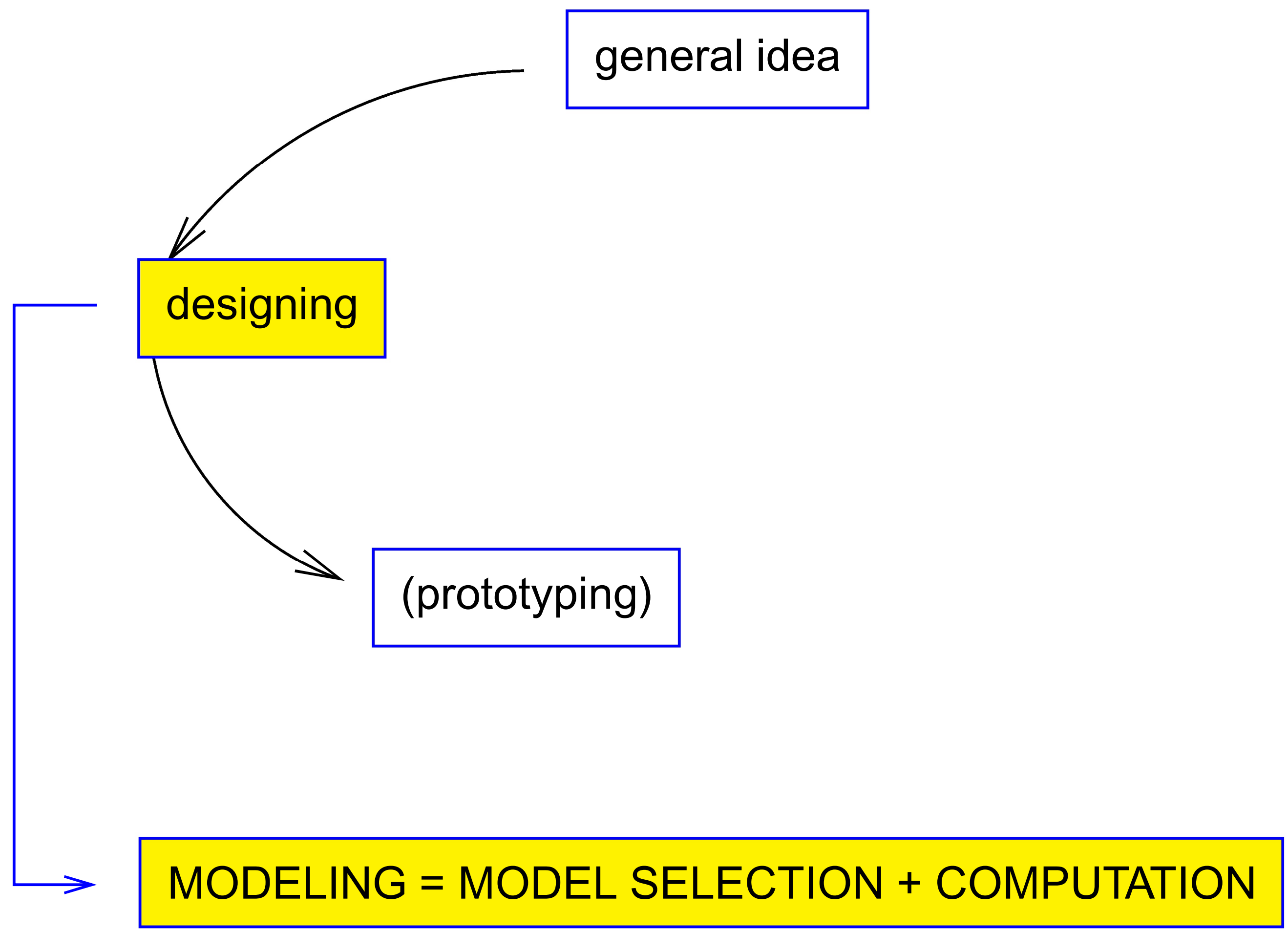
Designing and Modeling in Engineering



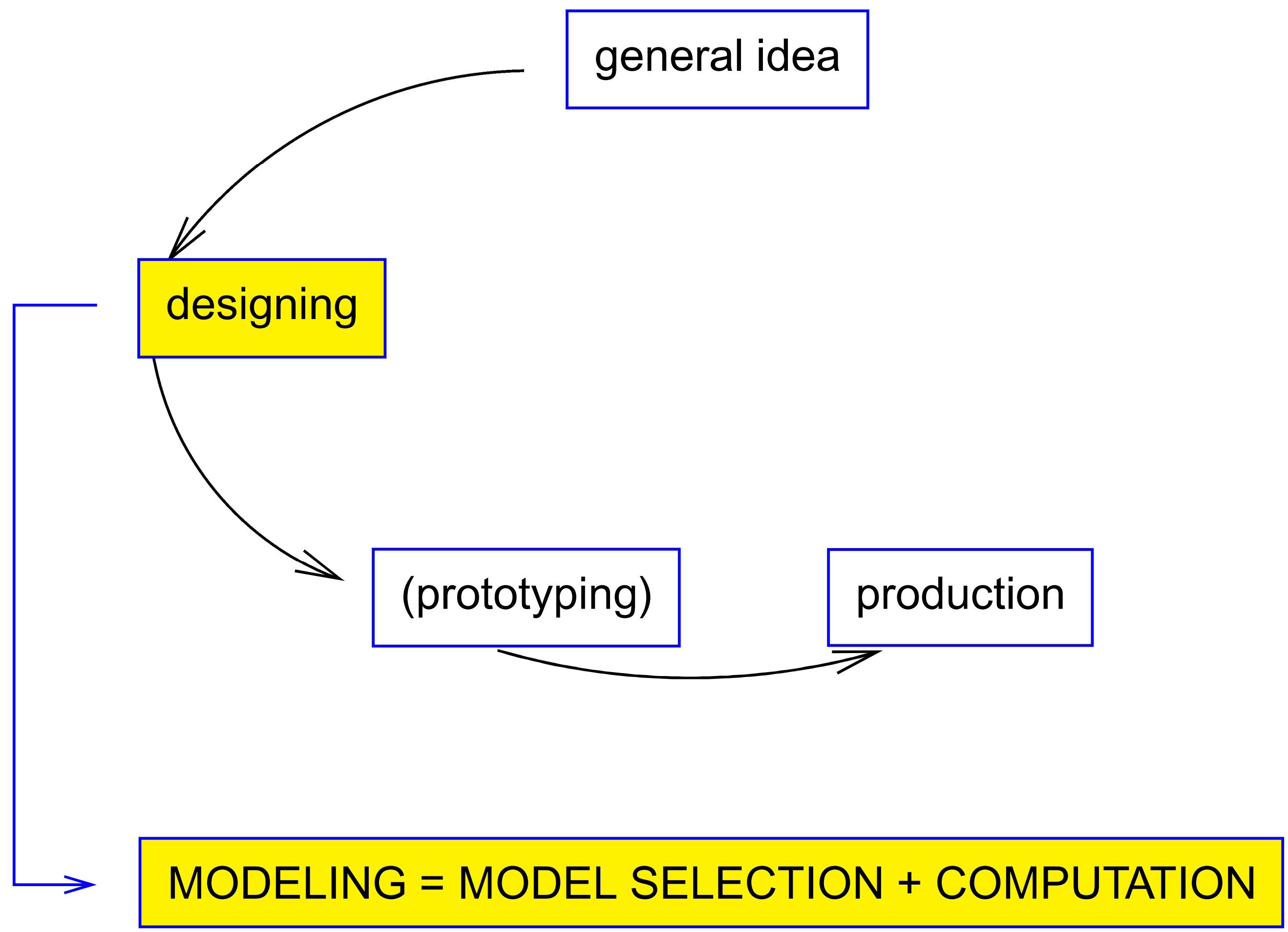
Designing and Modeling in Engineering



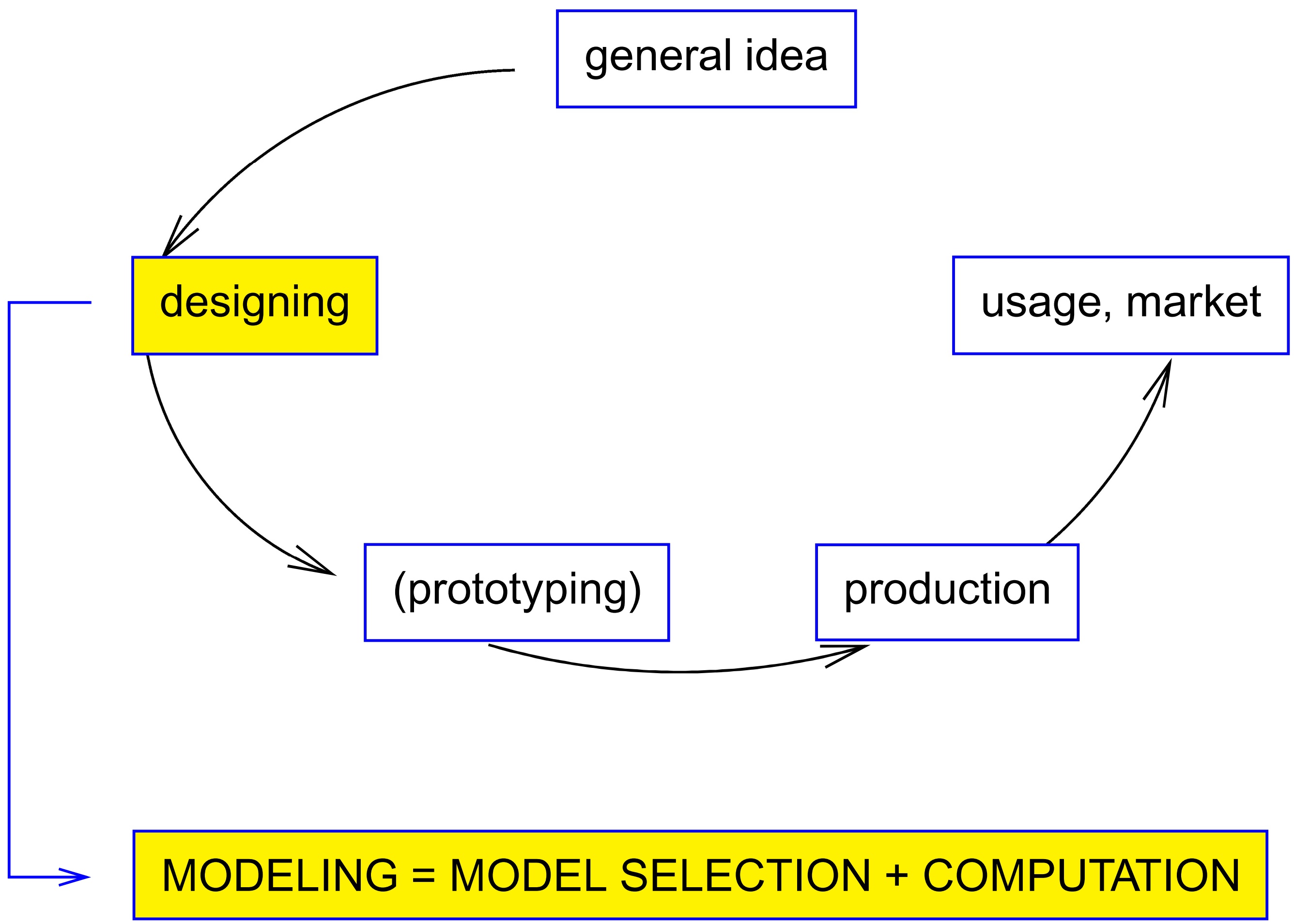
Designing and Modeling in Engineering



Designing and Modeling in Engineering



Designing and Modeling in Engineering



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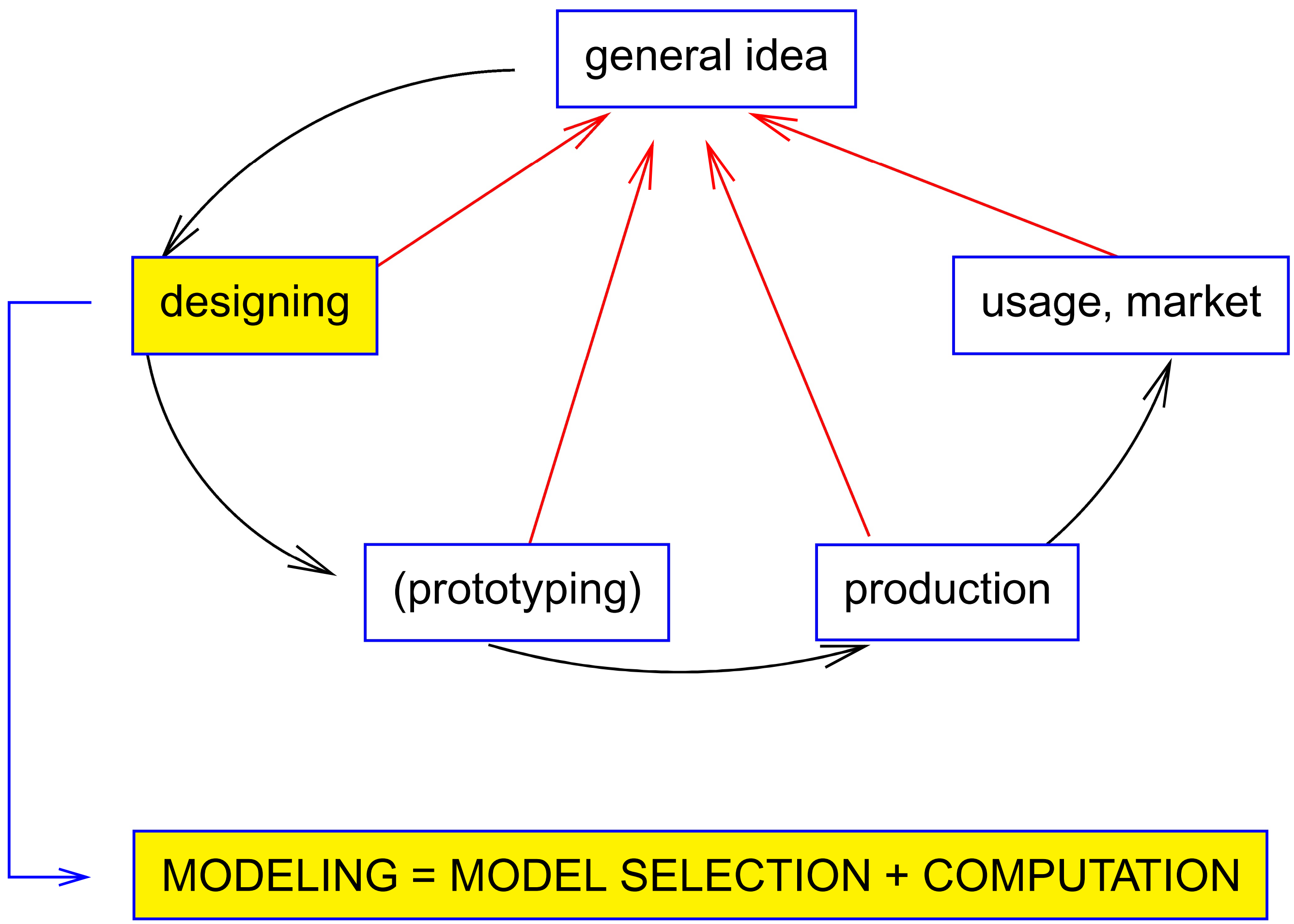

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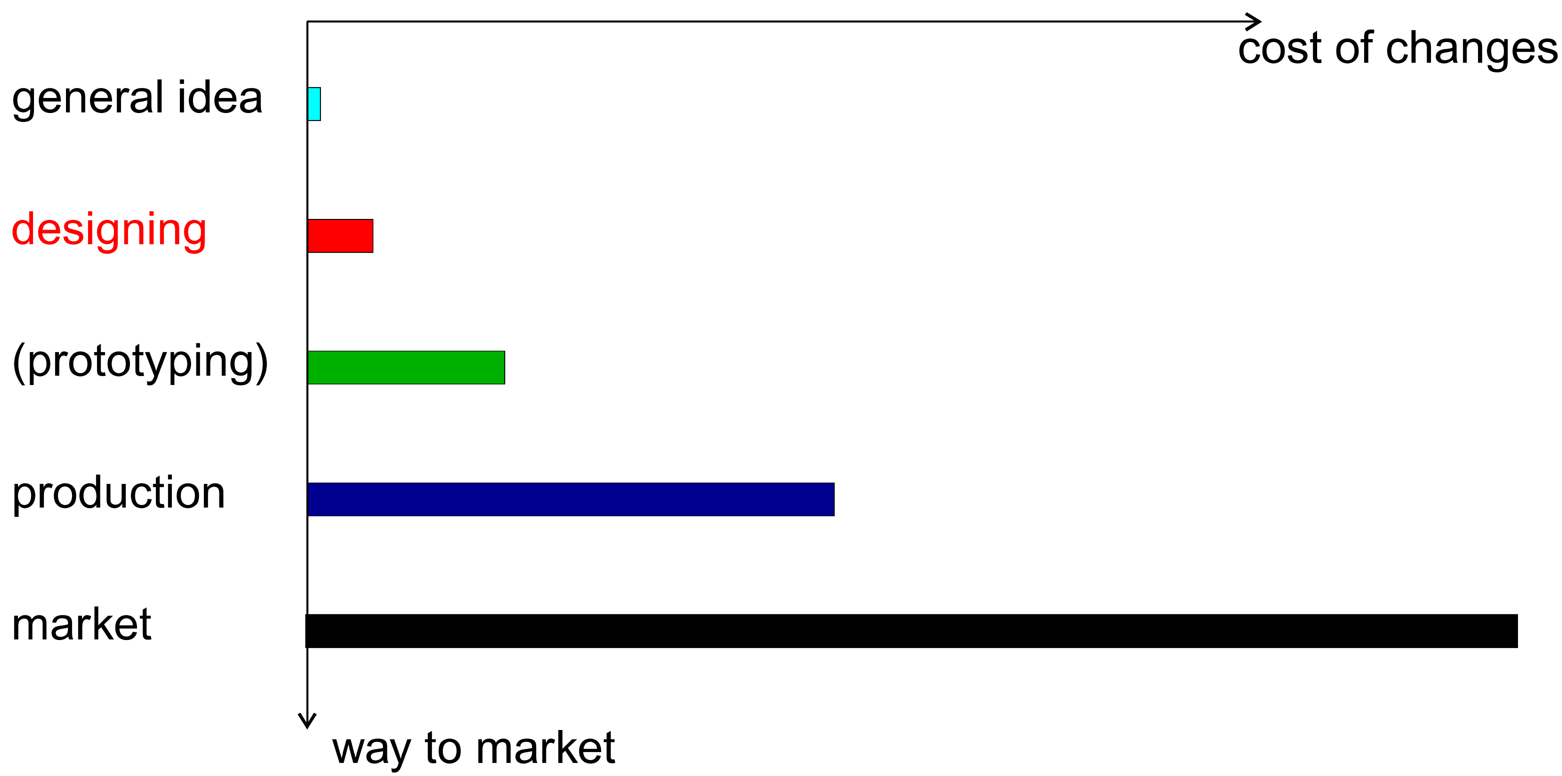

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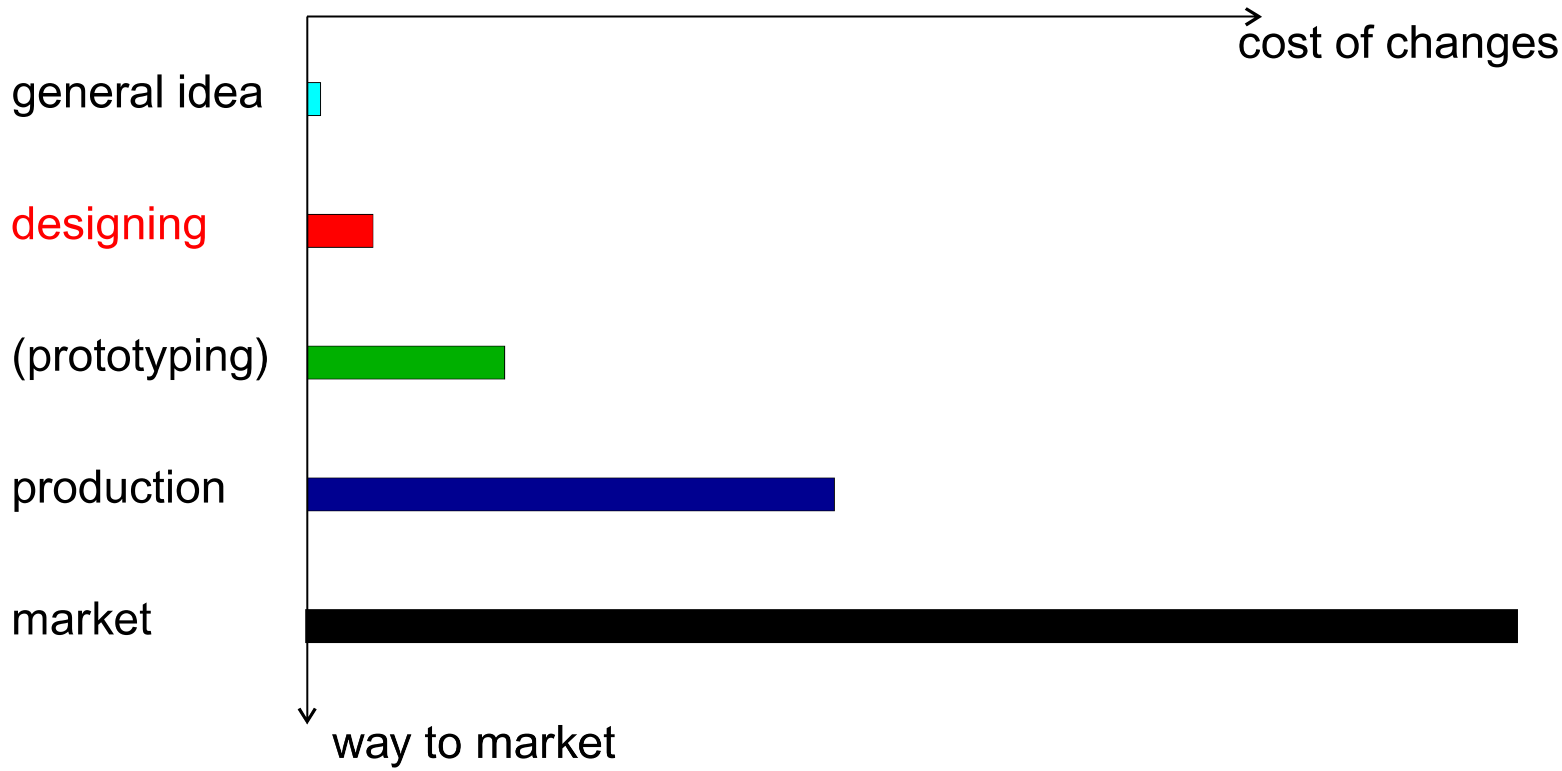


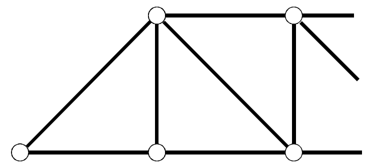
A recent study sponsored by the United States Government concluded that enterprise-wide “... modeling and simulation are emerging as key technologies to support manufacturing in the 21st century, and no other technology offers more potential than modeling and simulation for improving products, perfecting processes, reducing design-to-manufacturing cycle time, and reducing product realization costs...”

Design is IMPERFECT, TRADE-OFFS are required,

RISK must be ACCEPTED but MITIGATED

- Model selection for
  - object

→

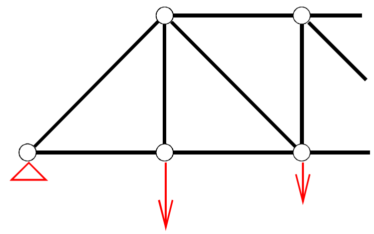


boundary value problem
(initial)

Modeling

- Model selection for
  - object + boundary conditions (+ initial conditions)

→



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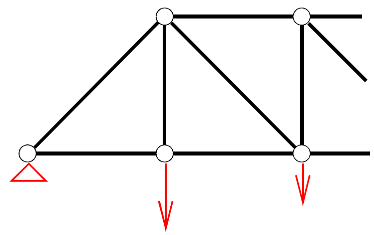
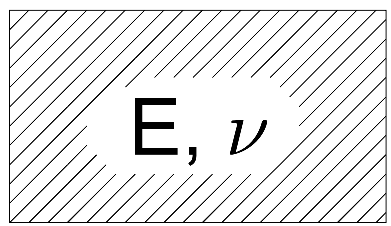


boundary value problem
(initial)

  - material



coefficients
(eg. constant)

Modeling

- Model selection for
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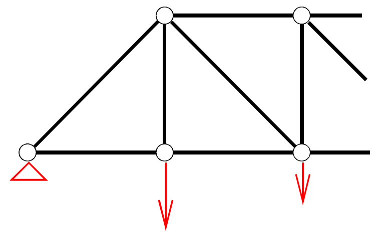

boundary value problem
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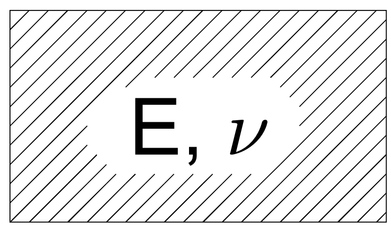


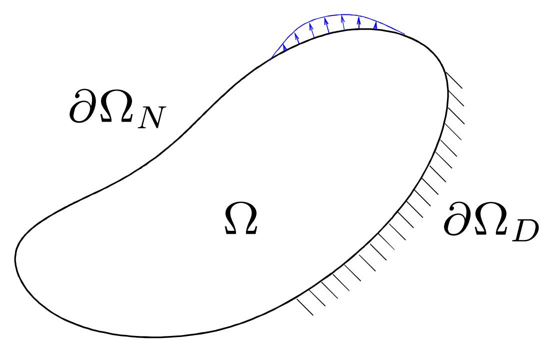
coefficients
(eg. constant)

  - values of parameters — *deterministic/stochastic* distribution

A Mathematical Model

- An example of a linear problem

  Find function u($\boldsymbol{x}$) $\in C^2(\Omega)$*:* $R^2 \ni \Omega \to R$ *such that*



$$\begin{cases} k\triangle u = -q & in \quad \Omega \\ u = 0 & on \quad \partial\Omega_D \\ k\dfrac{\partial u}{dn} = \hat{g} & on \quad \partial\Omega_N \end{cases}$$

or

$$u \in H_0^1; \quad \int_\Omega k\nabla v \circ \nabla u \,\mathrm{d}\Omega = \int_\Omega vq\,\mathrm{d}\Omega + \int_{\partial\Omega_N} v\hat{g}\,\mathrm{d}s \qquad \forall v \in H_0^1$$

- in general

  $L(u) = -q \ \ (+\, b.c.)$ or $b(v,u) = l(v) \qquad \forall v \in V$

FEM applications

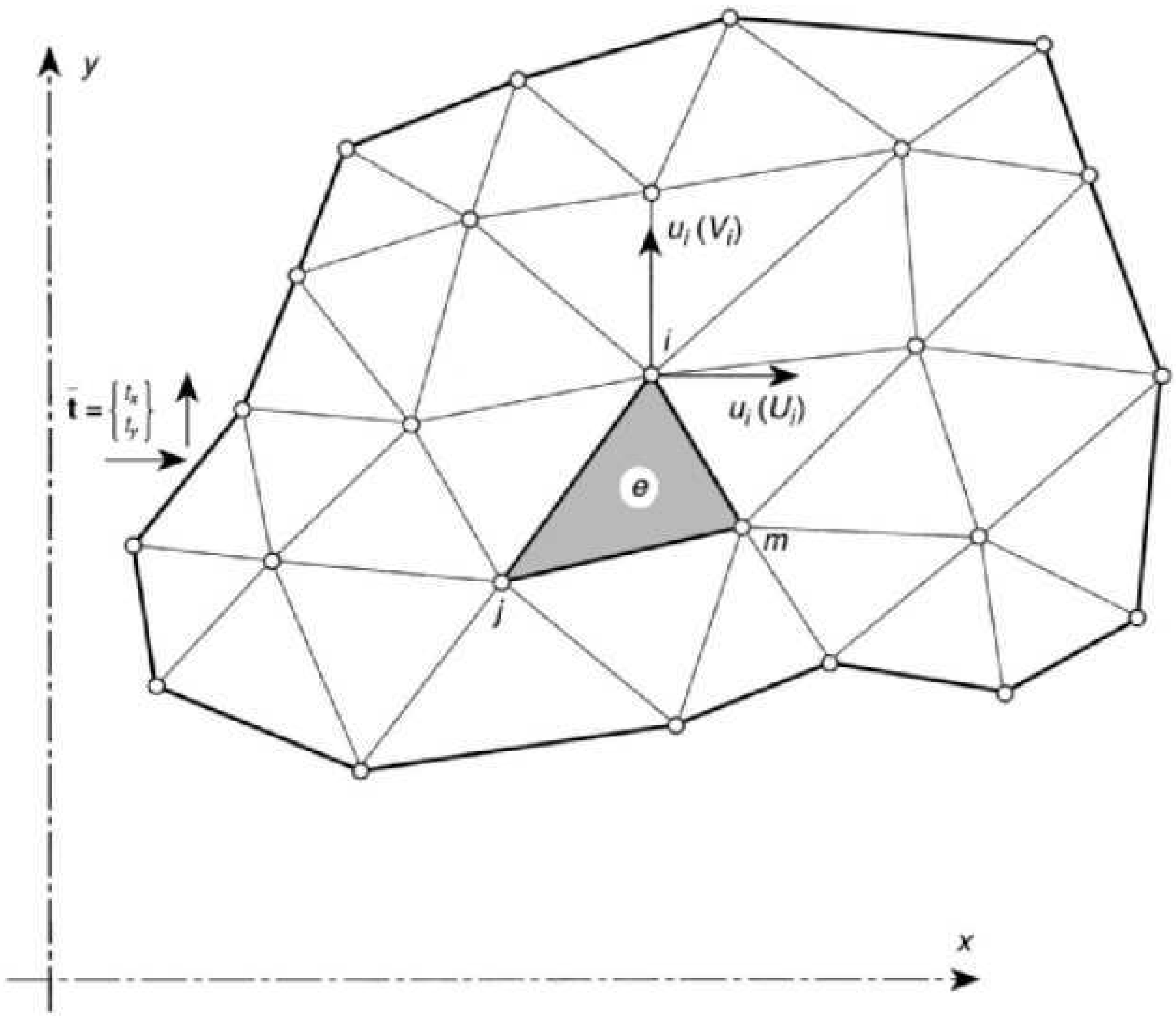


Fig. 2.1 A plane stress region divided into finite elements.

Shape functions

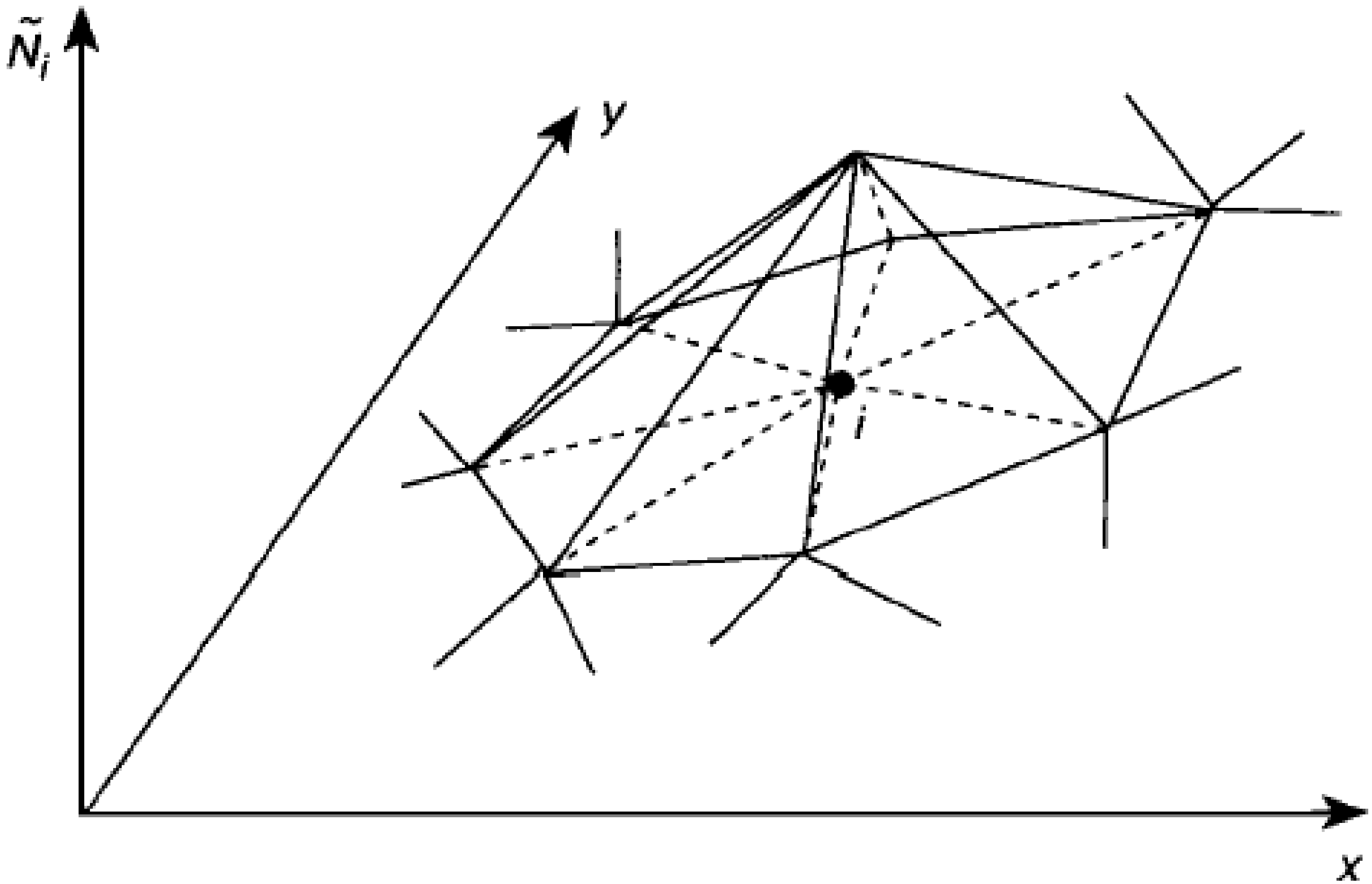
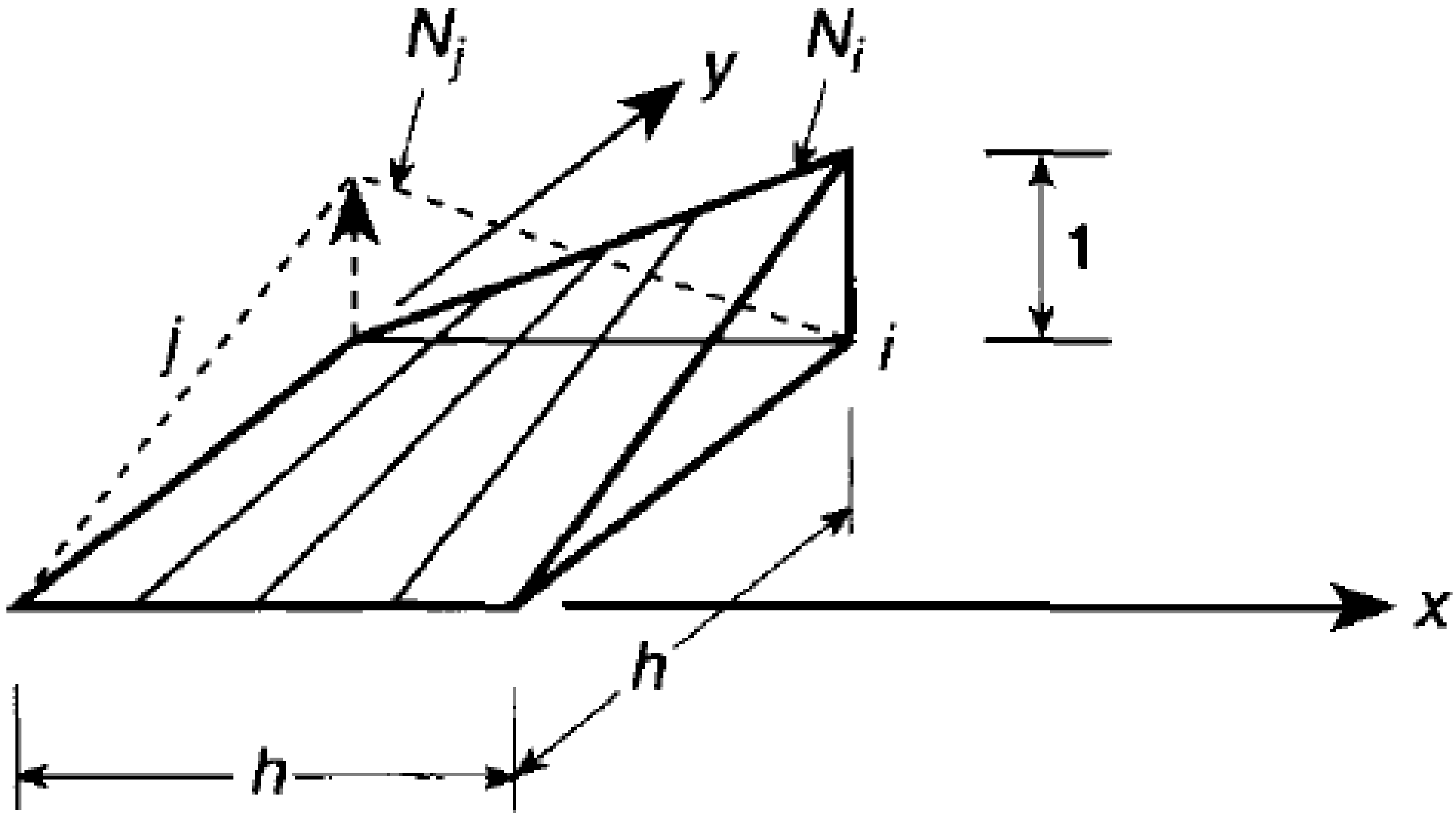


Fig. 2.3. A 'global' shape function – $\bar{N}_i$

Shape functions



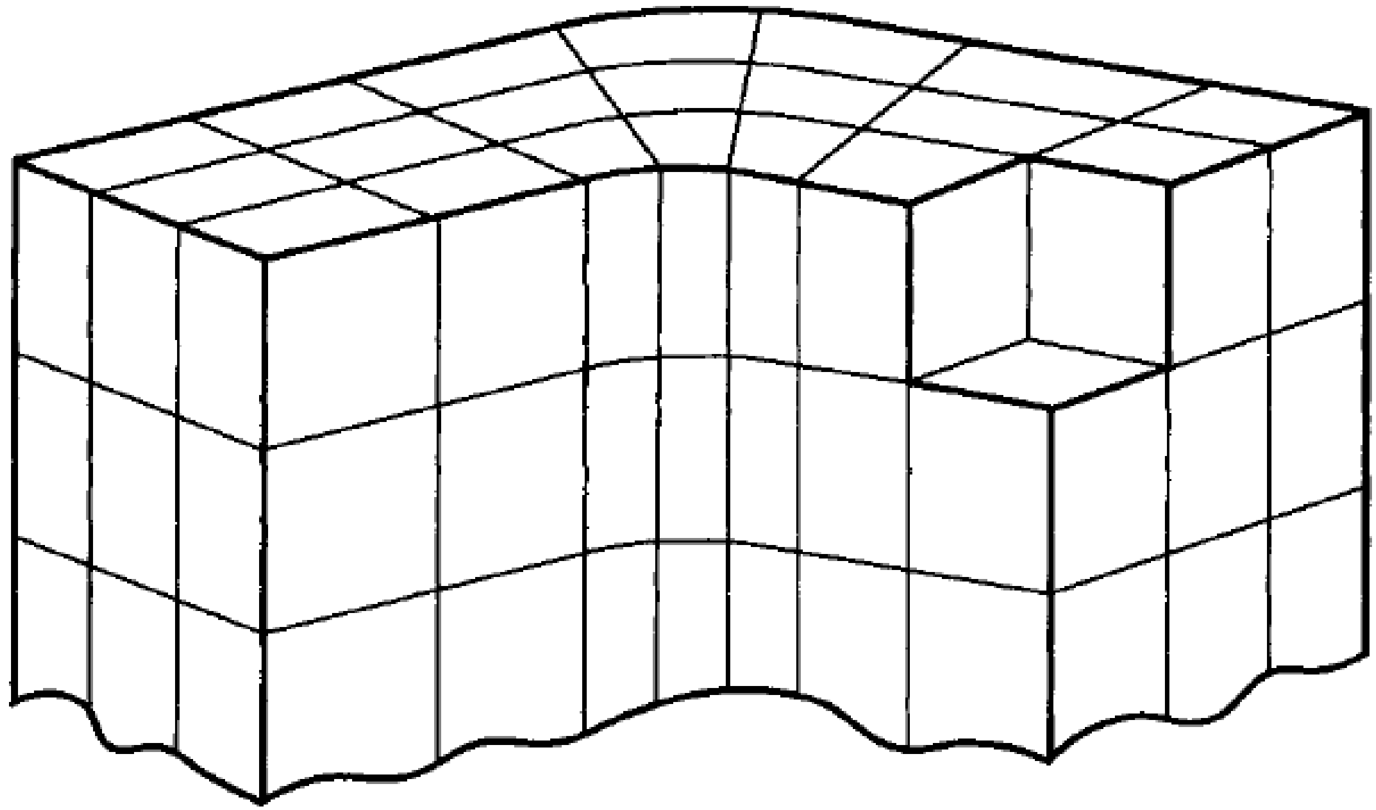


Fig. 6.2 A systematic way of dividing a three-dimensional object into 'brick'-type elements.

FEM applications

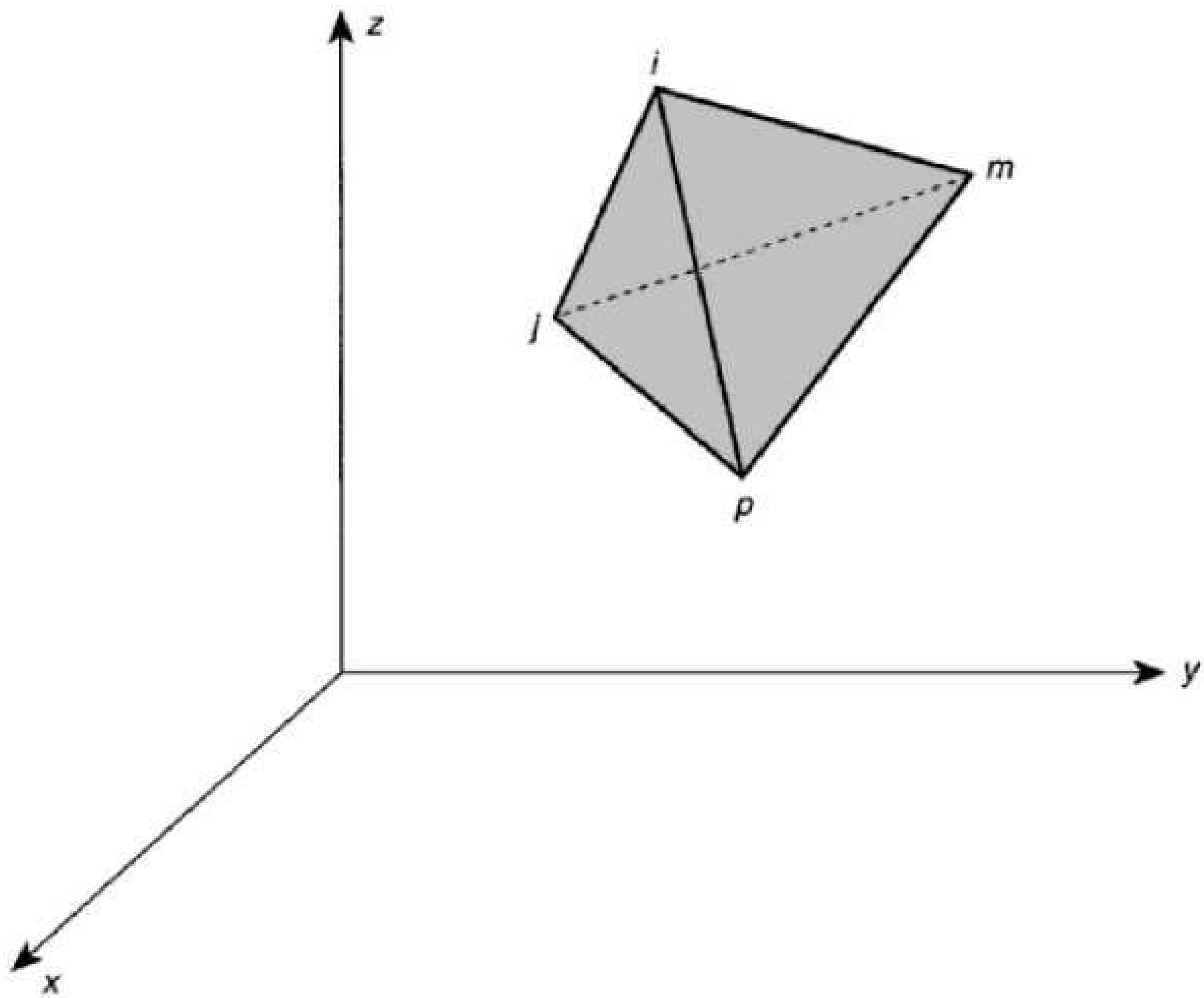


Fig. 6.1 A tetrahedral volume. (Always use a consistent order of numbering, e.g., for *p* count the other node: in an anticlockwise order as viewed from *p*, giving the element as *ijmp*, etc.).

- Solution Aproximation

  - basis functions $$u_X(x) = \sum_{i=1}^{N} \alpha_i \, \varphi_i(x)$$

Modeling

- Solution Aproximation
  - basis functions $$u_X(x) = \sum_{i=1}^{N} \alpha_i \, \varphi_i(x)$$
- Algorithm
  - cut-off errors   iterations, expansions ...

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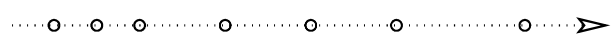
  - round-off errors 

R_{comp} is not closed with respect to +,-,*,/ operations

Modeling

- Solution Aproximation
  - basis functions $$u_X(x) = \sum_{i=1}^{N} \alpha_i \, \varphi_i(x)$$
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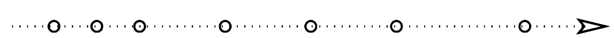

R_{comp} is not closed with respect to +,-,*,/ operations

$$\sqrt{x+1} - \sqrt{x} = \frac{1}{\sqrt{x+1} + \sqrt{x}}$$

Modeling

- Other error sources
  - Insufficient user knowledge
    inadequate model
    inappropriate mesh
    improper result interpretation
  - Bug in the code
  - Wrong data
  - • • •

Modeling

- Other error sources
  - Insufficient user knowledge
    inadequate model
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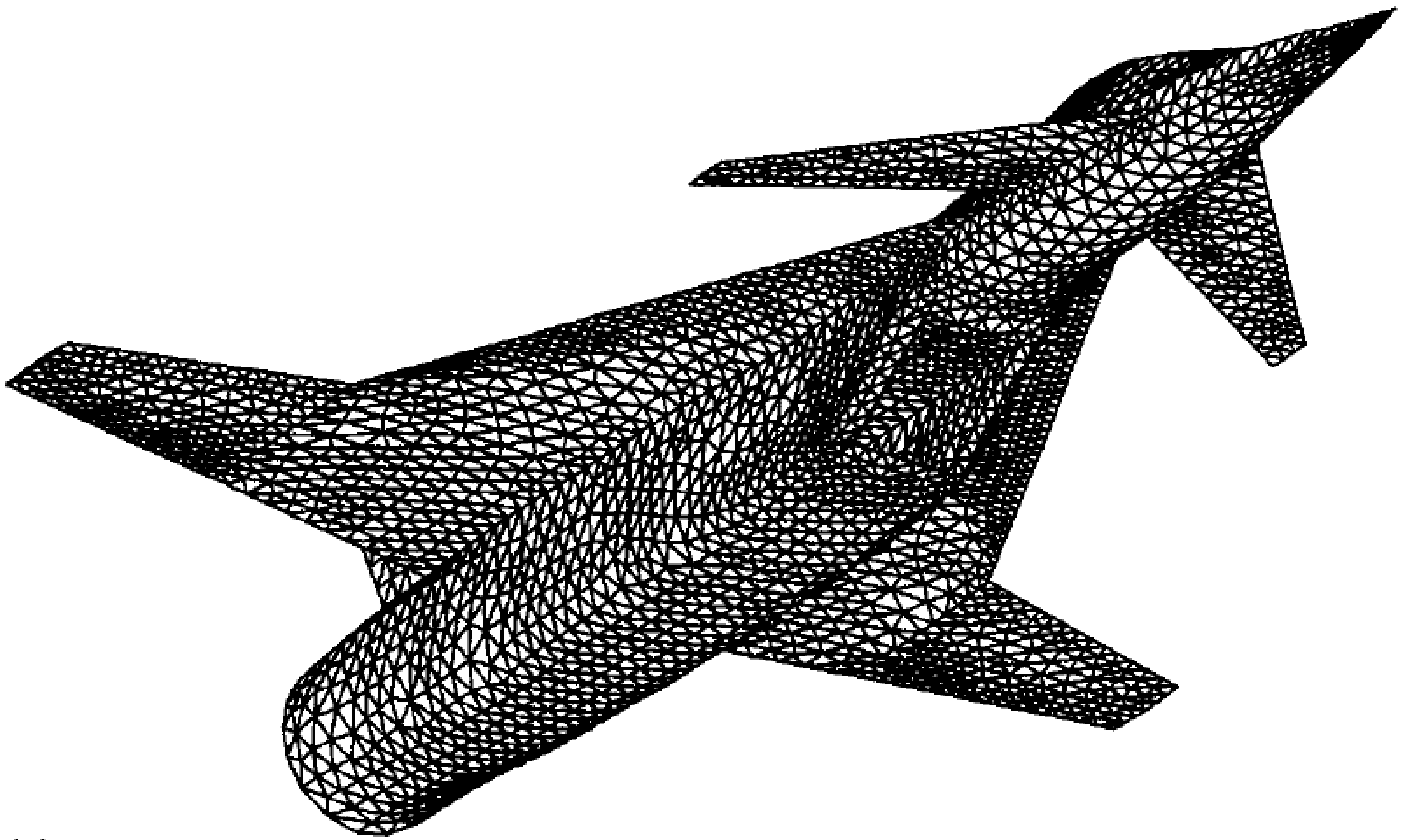
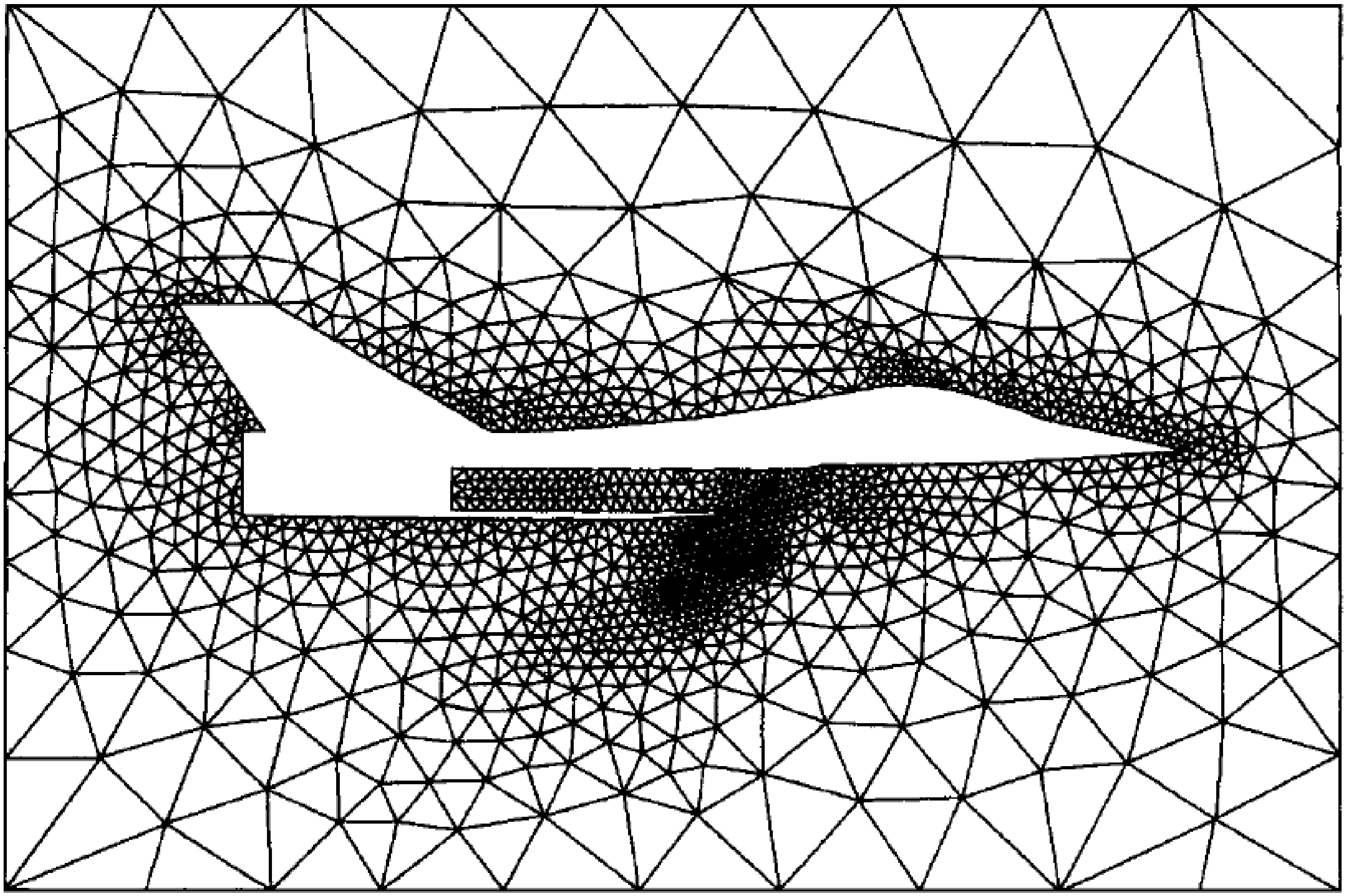
  - Bug in the code
  - Wrong data
  - • • •
- Mathematics in modeling
  - If we are not sure that a solution exists then what we try to approximate numerically?
  - If we do not know which class of functions the solution belongs to, then we cannot properly define its approximation and the measure for the accuracy
  - Classical error control theory is mainly focused on approximation errors

FEM applications



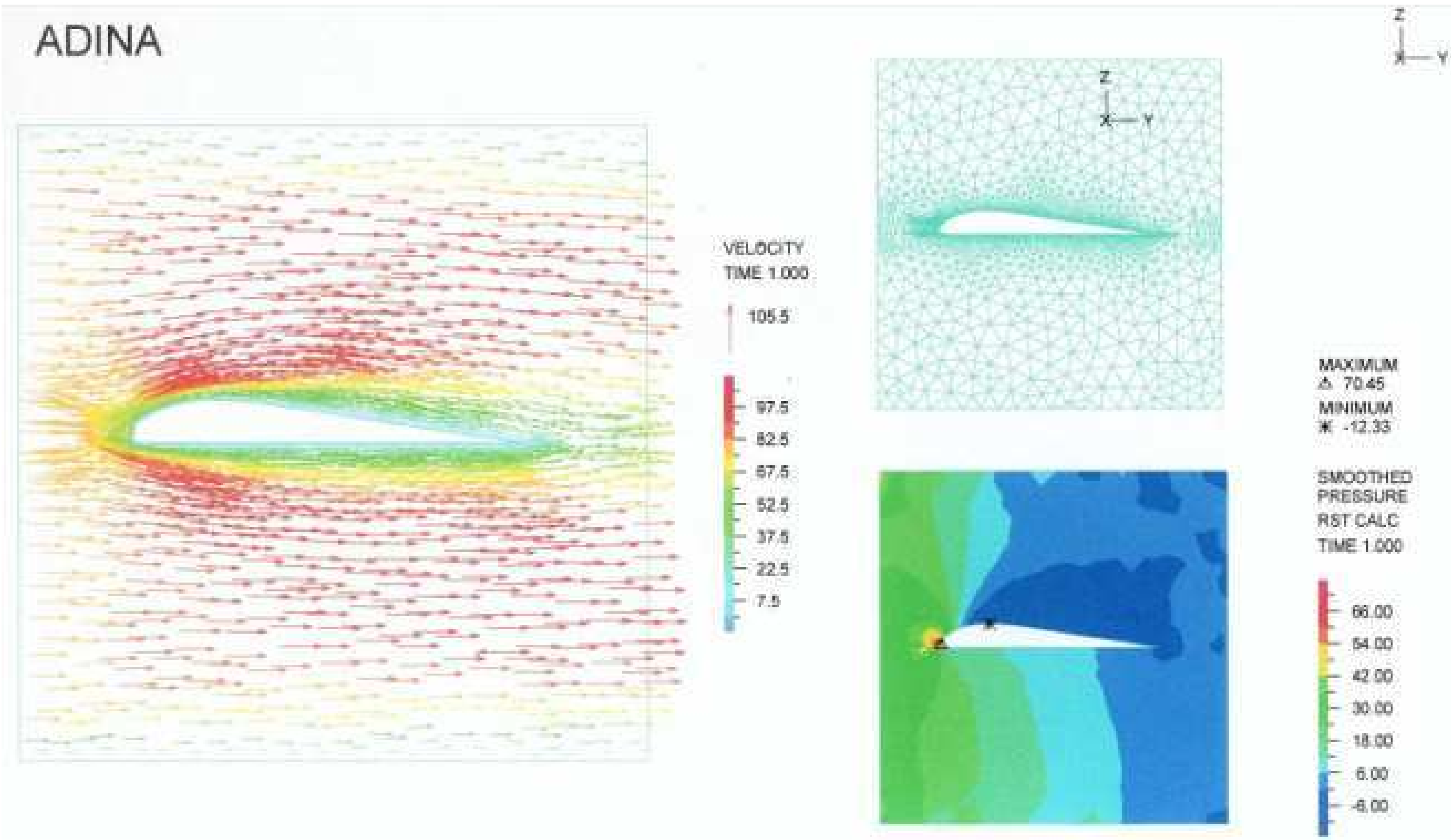
(a)

FEM applications



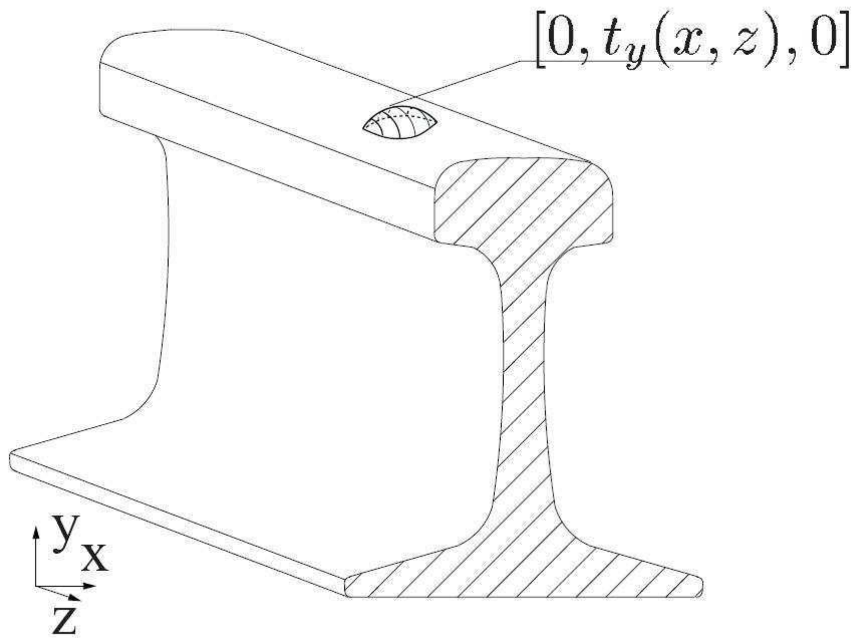
(b)

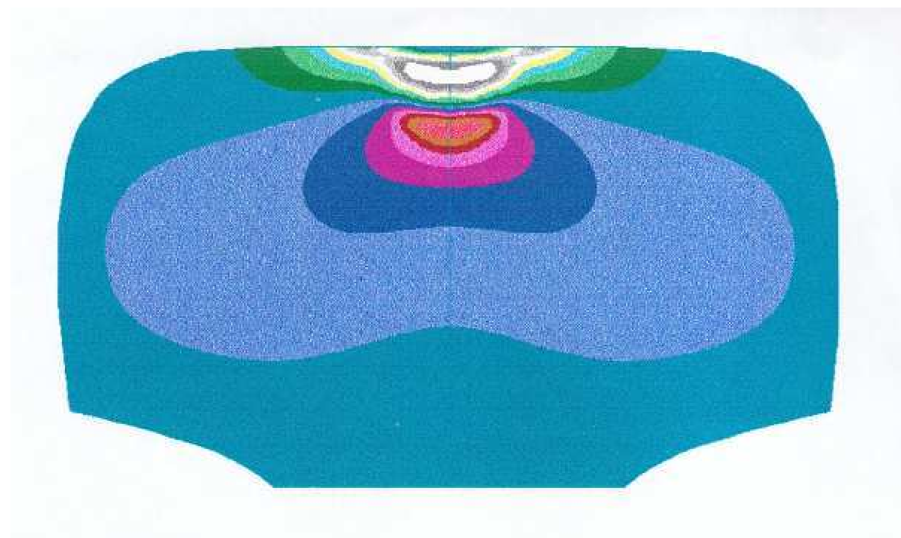
FEM applications



Air flow around an airplane wing

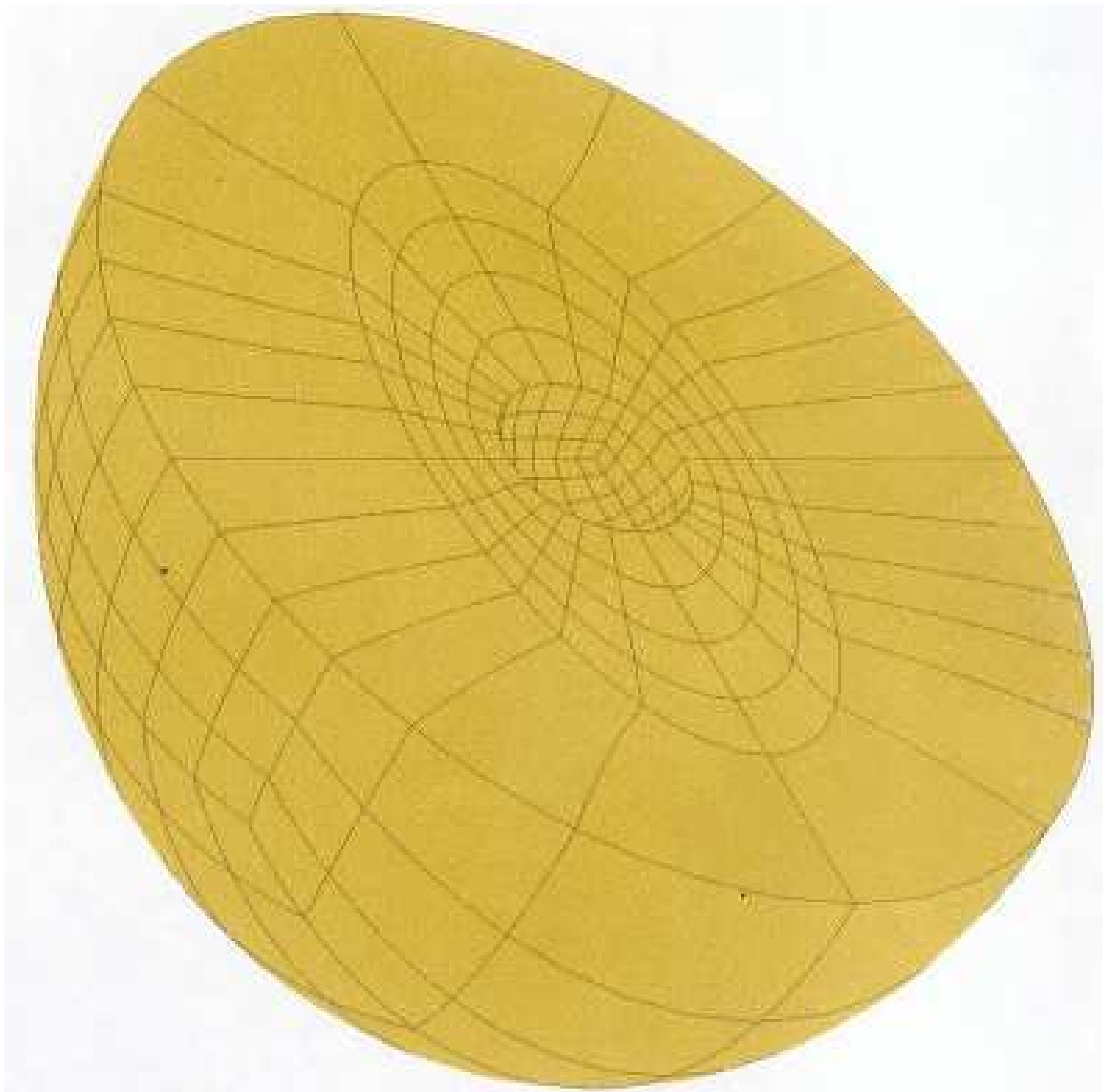
FEM applications





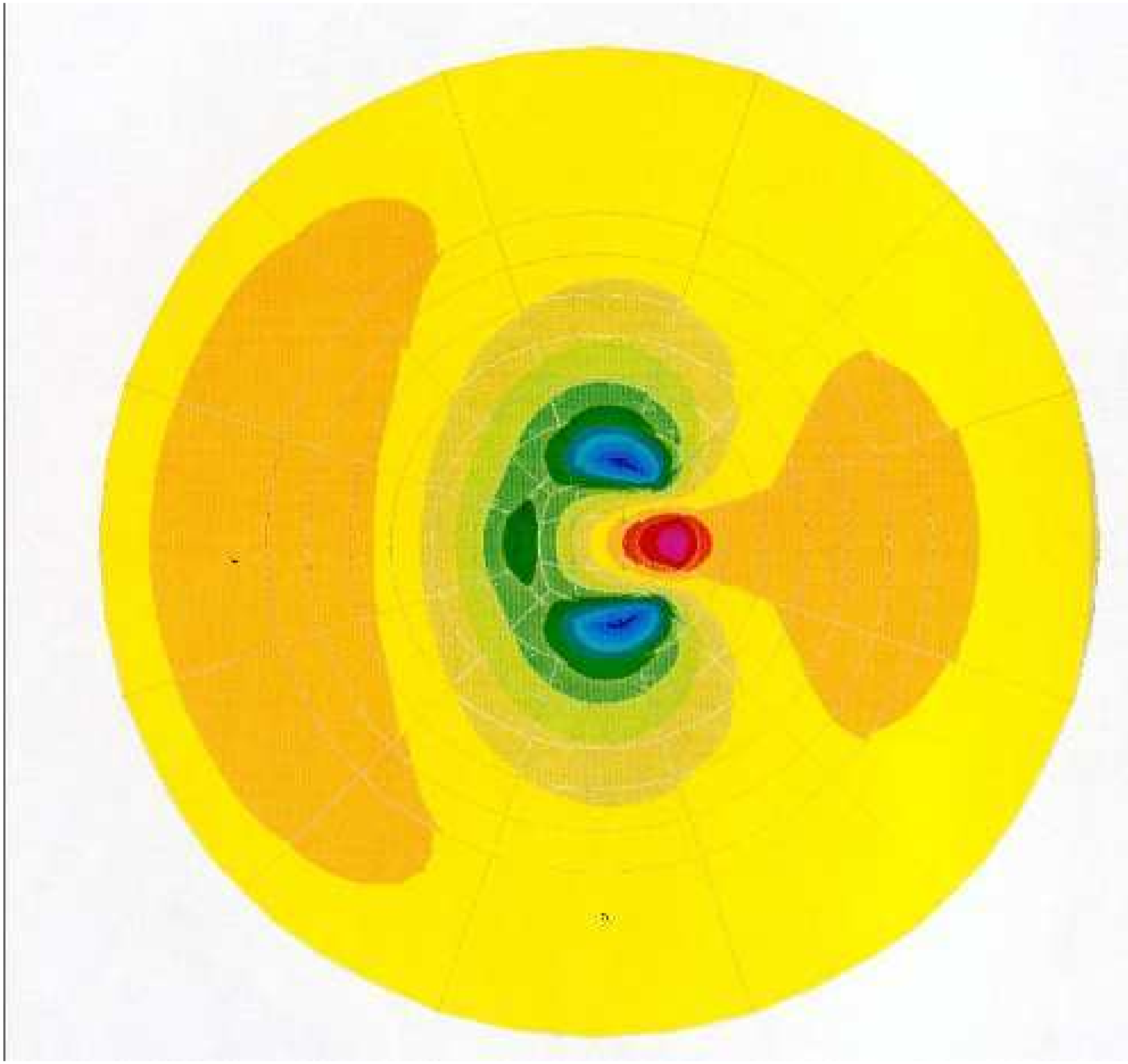
Longitudinal residual stress component in railroad rail

FEM applications



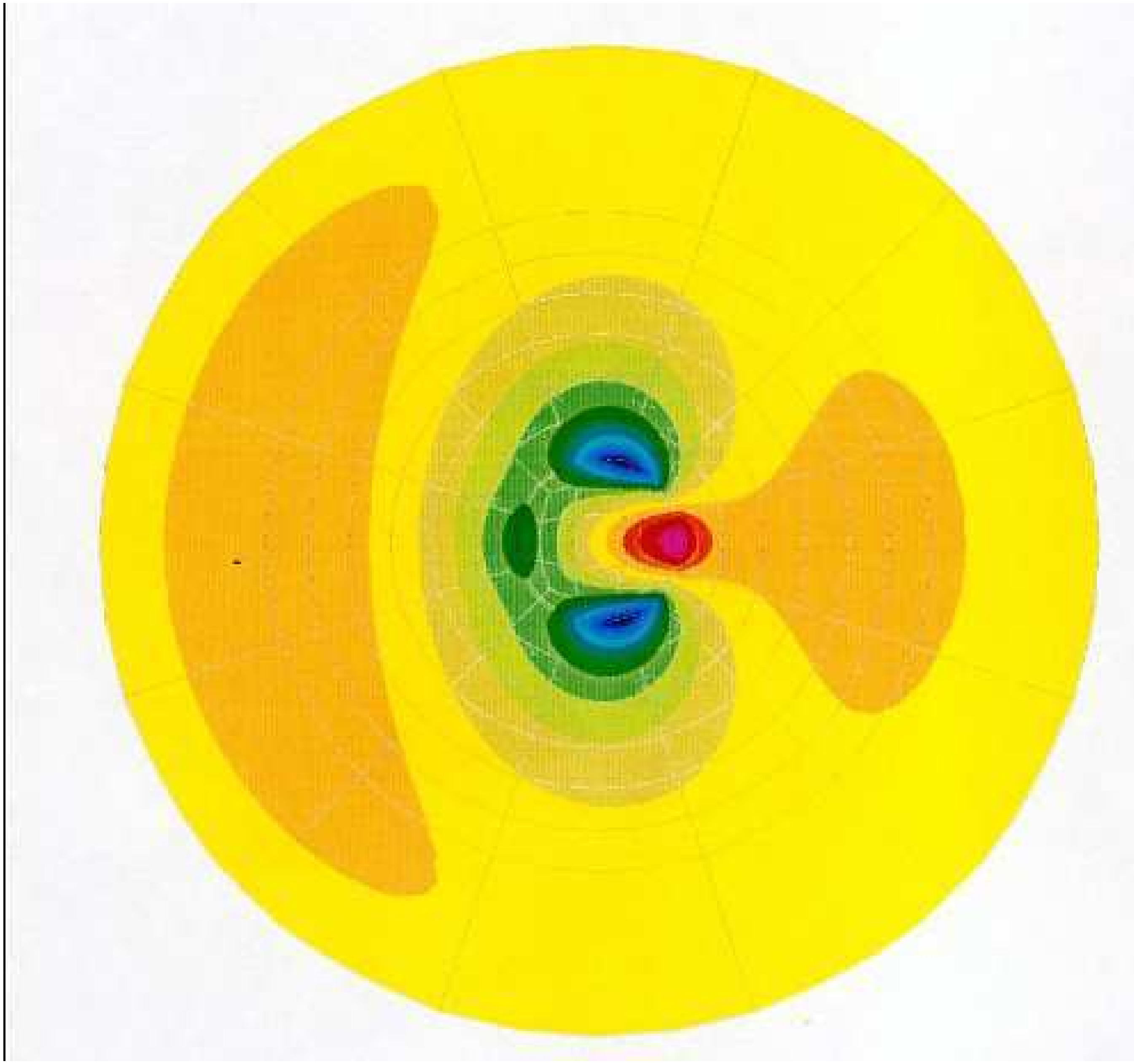
Scattering of electromagnetic waves
Exterior of a ball discretized by finite and infinite elements

FEM applications



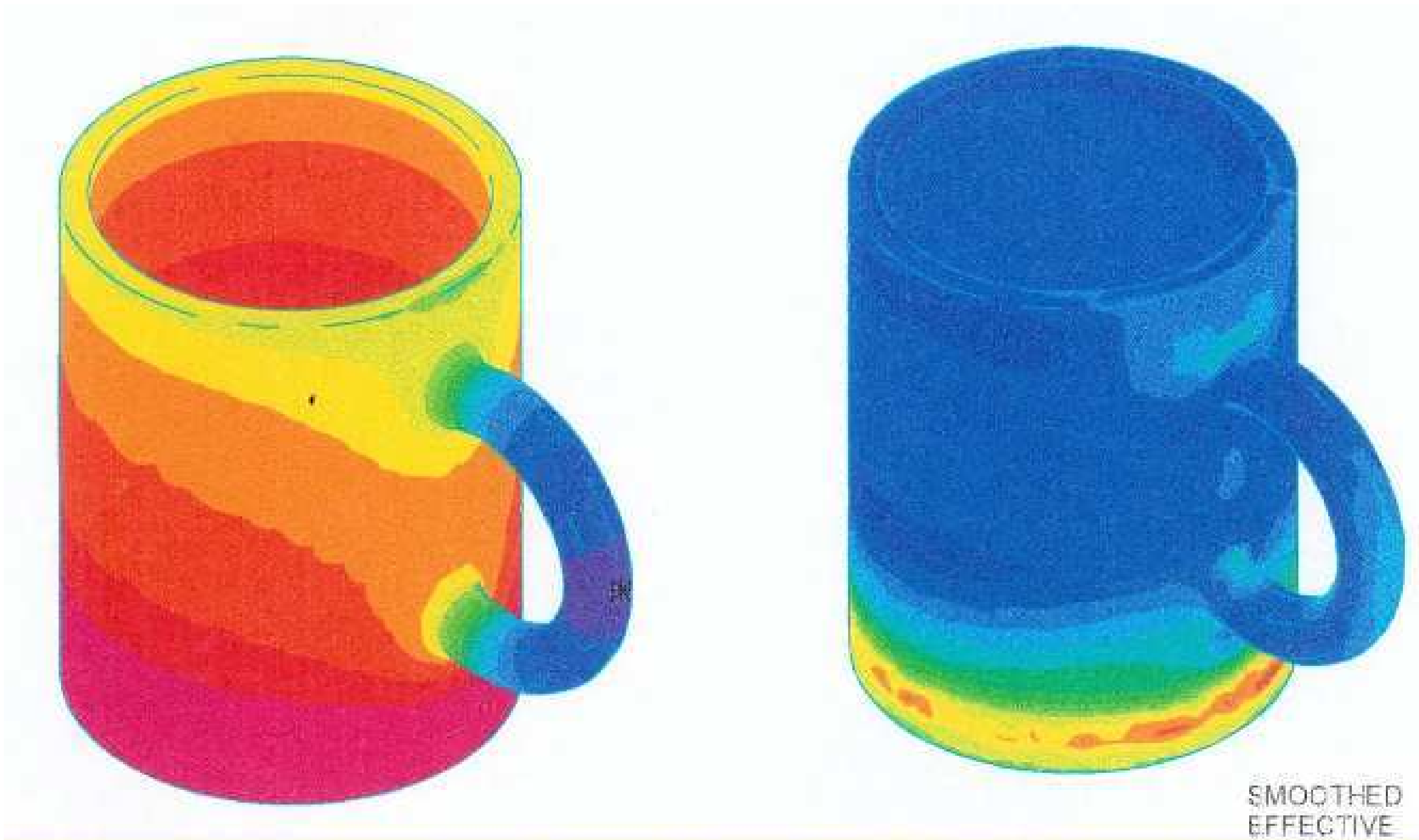
FEM approximation of electric field

FEM applications



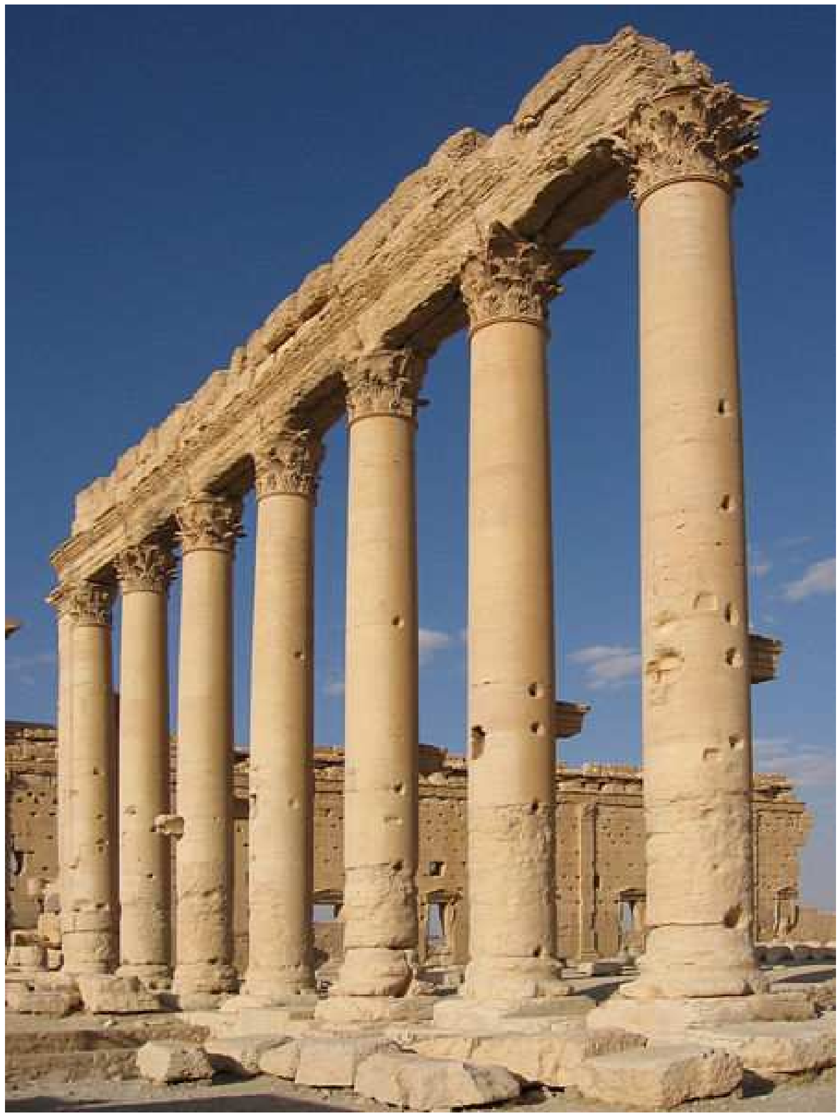
Exact electric field

FEM applications

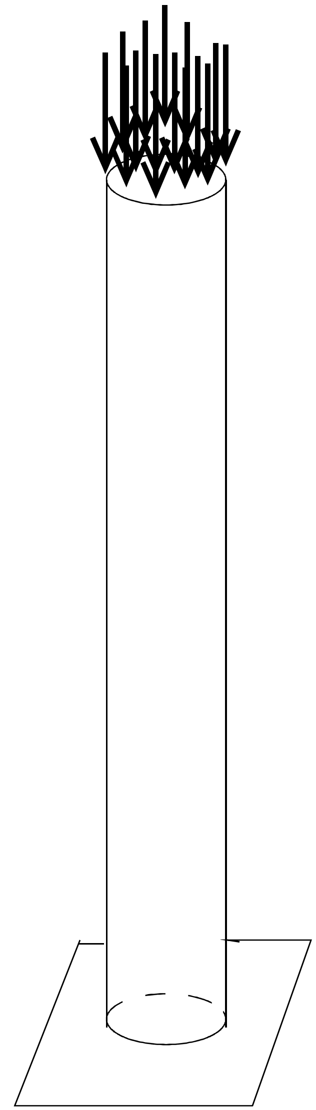


Heat transfer

Modeling

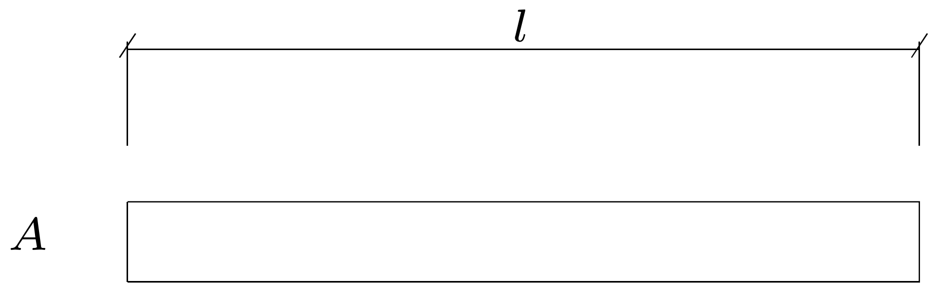


Columns in Syria

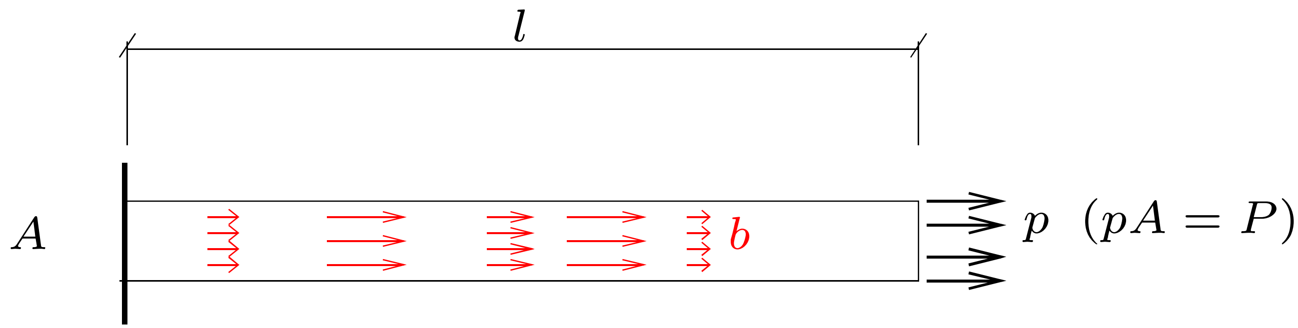


Model

Problem formulation



Problem formulation



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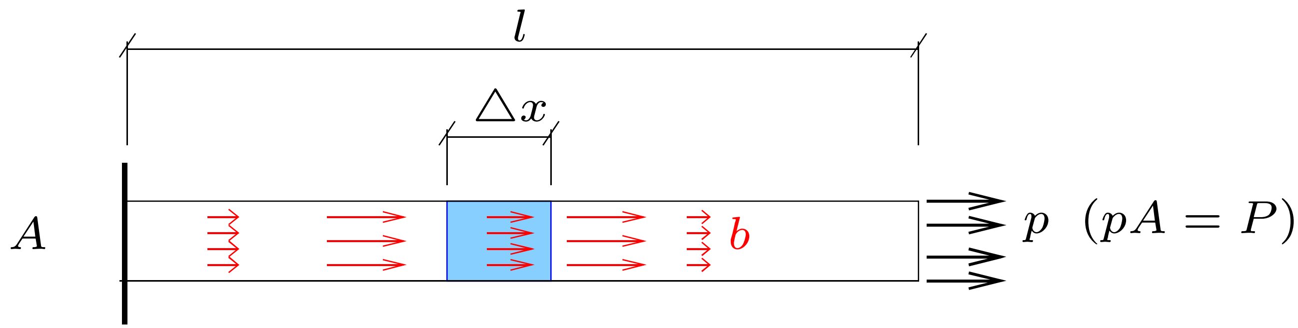

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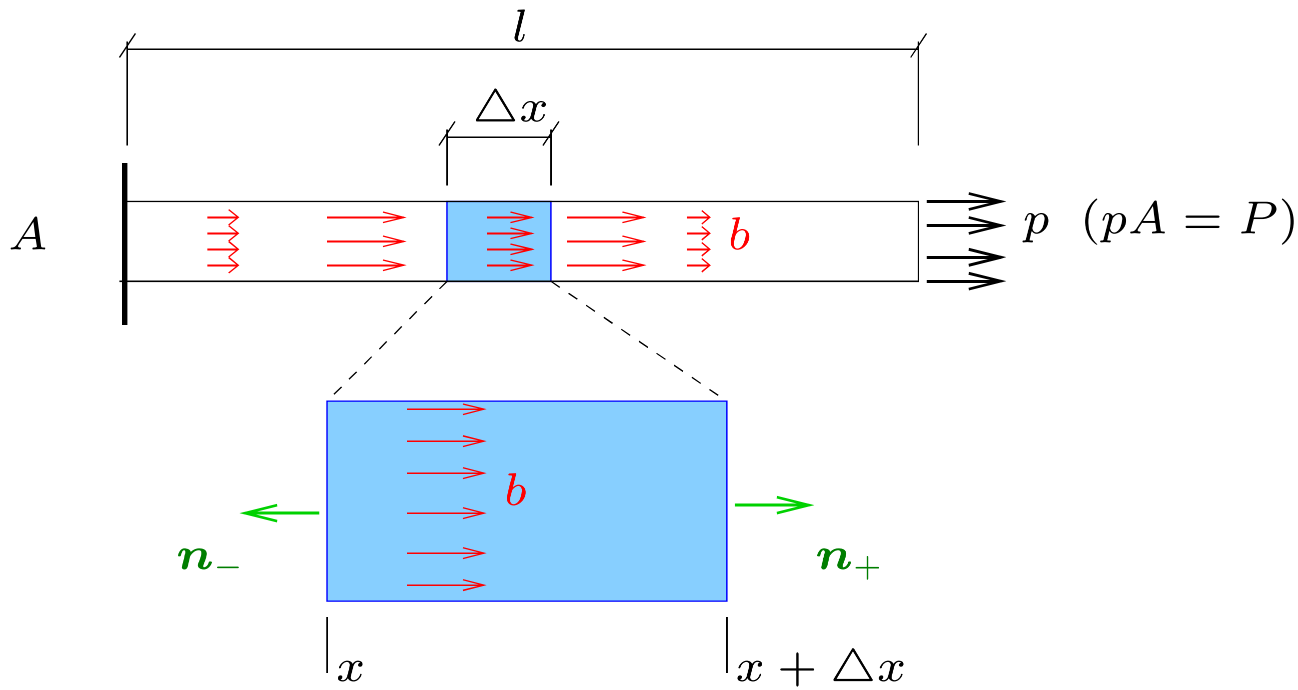

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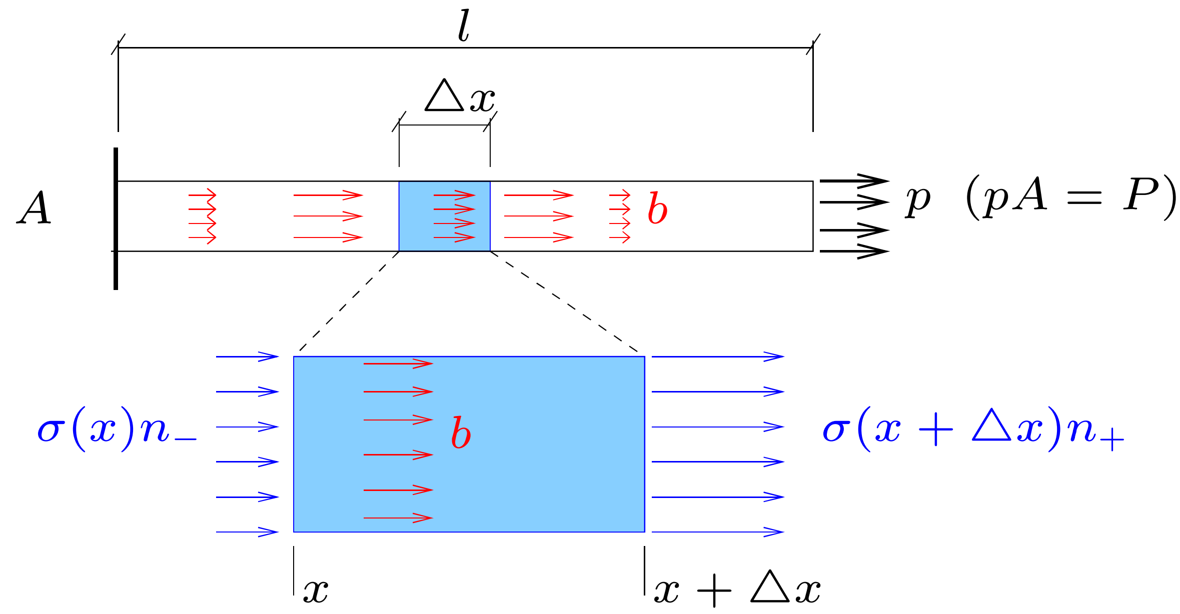

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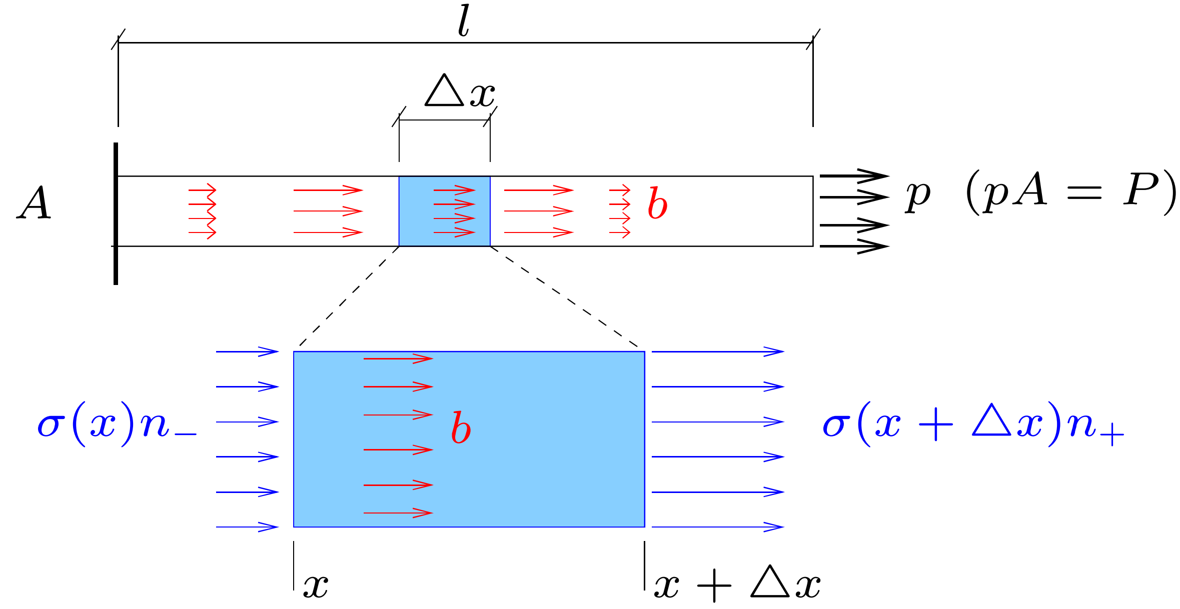


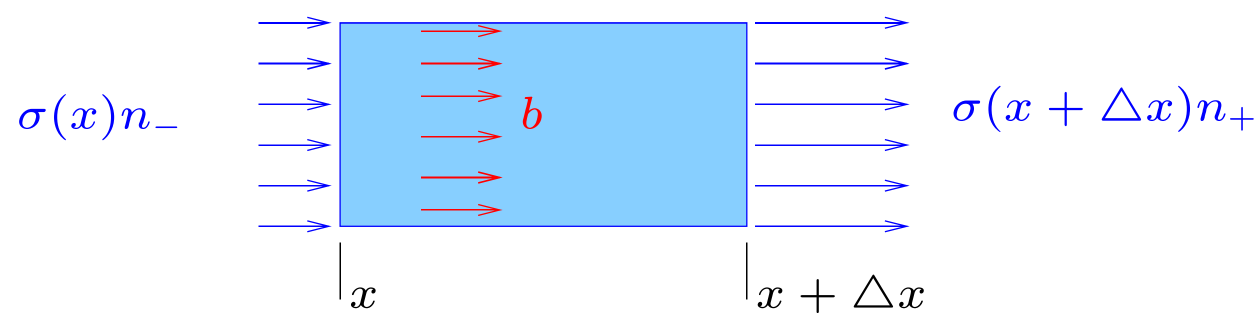
$\sigma = \sigma(x), \quad b = b(x) \quad \longrightarrow \quad \sigma(x)A = N(x), \quad b(x)A = q(x)$

elastic material $\longrightarrow \quad \sigma(x) = E\varepsilon(x)$

small displacement gradients $\longrightarrow \quad \varepsilon(x) = \dfrac{du}{dx} \quad \longrightarrow \quad \sigma = E\dfrac{du}{dx}$

short range of intermolecular forces

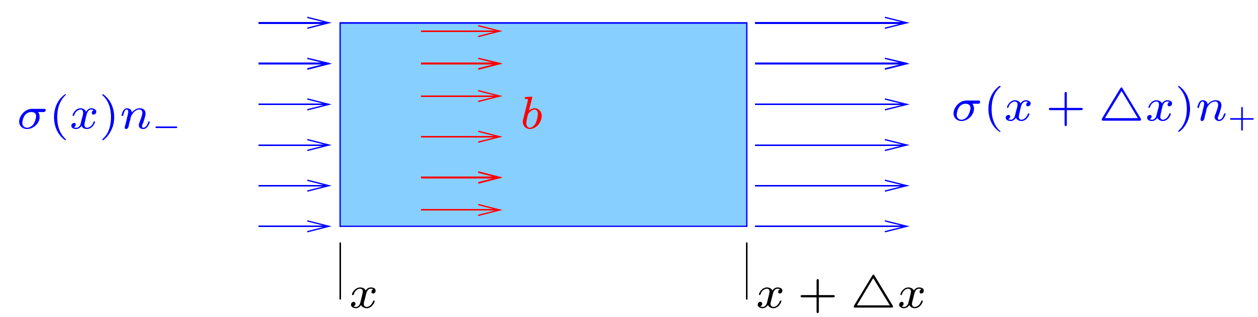
Problem formulation



Momentum Conservation Principle (Second Newton's Law of Motion)

$\longrightarrow$ Equilibrium Equations

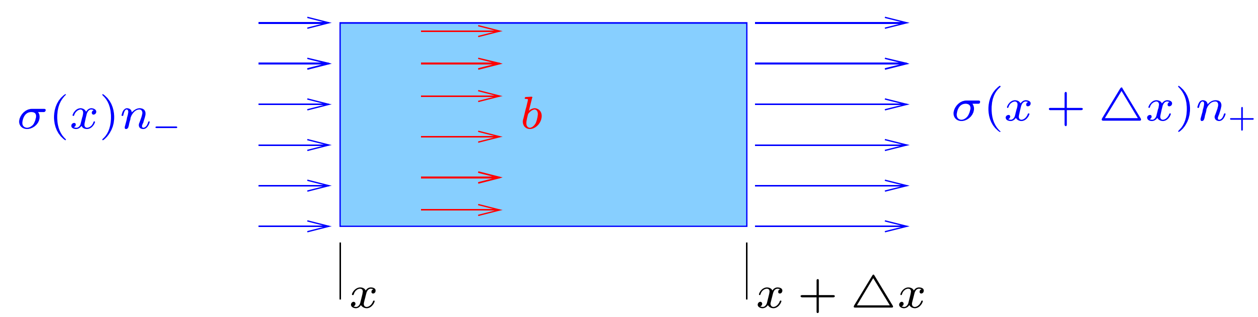
Problem formulation



Momentum Conservation Principle (Second Newton's Law of Motion)
$\longrightarrow$ Equilibrium Equations

$$A\sigma(x)n_- + \int_x^{x+\triangle x} q(y)\,\mathrm{d}y + A\sigma(x+\triangle x)n_+ = 0 \qquad \forall \omega \subset (0,l)$$

Problem formulation



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$$n_- = -1, \quad n_+ = 1$$

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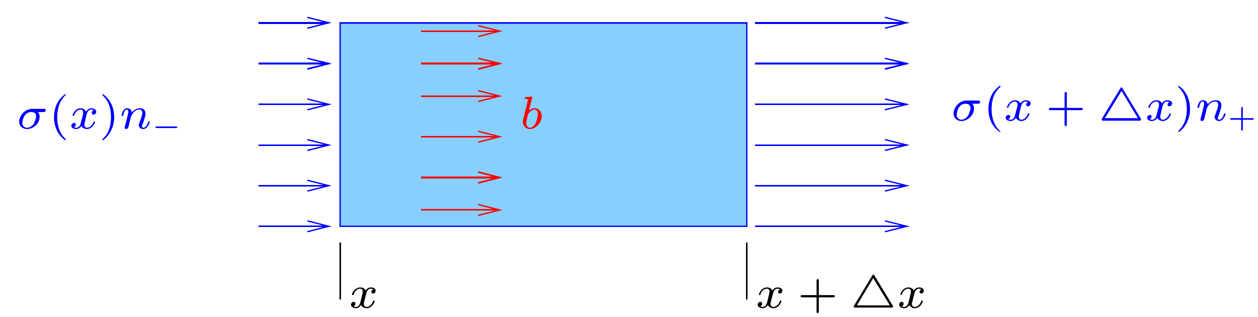

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$$A\sigma(x)n_- + \int_x^{x+\triangle x} q(y)\,\mathrm{d}y + A\sigma(x+\triangle x)n_+ = 0 \qquad \forall\omega\subset(0,l)$$

$$n_- = -1, \quad n_+ = 1$$

Find u(x) such that:

$$AE\frac{du}{dx}(x+\triangle x) - AE\frac{du}{dx}(x) = -\int_x^{x+\triangle x} q(y)\,\mathrm{d}y \qquad \forall\omega\subset(0,l) + \text{b.c.}$$

$\rightarrow$ FVM

Problem formulation

- Taylor formula: $\exists \xi : \frac{du}{dx}(x+\triangle x) = \frac{du}{dx}(x) + \frac{d^2u}{dx^2}(\xi)\triangle x$ (if u'' exists)
- Mean value theorem: $\exists \eta : \int_x^{x+\triangle x} q(y)\,\mathrm{d}y = q(\eta)\triangle x$ (if q is continuous)

Problem formulation

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- $\triangle x \to 0$

Find $u(x) \in C^2([0,l])$ such that:

$$\begin{cases} AE\frac{d^2u}{dx^2} = -q(x) \qquad \forall x \in (0,l) \\ u(0) = 0 \\ AE\frac{du}{dx}(l)n(l) = P \end{cases}$$

$\rightarrow$ FDM

Thank you