1 Motivation

US Government study: "... modeling and simulation are emerging as key technologies to support manufacturing in the 21st century, and no other technology offers more potential than modeling and simulation for improving products, perfecting processes, reducing design-to-manufacturing cycle time, and reducing product realization costs...”.

The Finite Element Method (FEM), which is the most popular modeling tool, is a very successful story. It is a way that engineers invented to solve equations of mechanics to find displacements and stresses in structures. The history of FEM begins in 1915 with B. Galerkin who used 2-3 sophisticated trial functions. In 40’s A. Hrennikoff and R. Courant had two different conceptions but the common key point to use many simple trial functions. First developments of the method are associated with the following names: J. Argyris, M. Turner, R. Clough (who coined the terminology finite element method) and O. Zienkiewicz.

Mathematicians contributed to understanding as well as the reliability of FEM. For many problems, FEM analysis is an art.

- FEM makes a good engineer great, and a bad engineer dangerous (R.Cook).
- Modeling (simulating nature) gives us insight into the world we live in.
- An engineer has to know how to: assume a model, solve it on a laptop, and assess the results.
## 2 General idea of the finite element method

1. Let’s consider the Laplace PDE equation (as a model of e.g. stationary heat transfer)

\[
\begin{align*}
  u(x, y) &\in C^2(\Omega) : \quad R^2 \supset \Omega \rightarrow R \\
  -k\Delta u &= f(x, y) \quad \text{in } \Omega \\
  u &= \hat{u} \quad \text{on } \partial\Omega_D \\
  k\frac{\partial u}{\partial n} &= \hat{t} \quad \text{on } \partial\Omega_N
\end{align*}
\]

(1)

2. Weak formulation ("virtual work principle") of the Laplace problem:

*Find continuous* \( u(x, y) \in H^1(\Omega) + \hat{u} \), *such that* \( u = \hat{u} \) *on* \( \partial\Omega_D \) and

\[
\int_\Omega k\nabla v \cdot \nabla u \, dx \, dy = \int_\Omega vf \, dx \, dy + \int_{\partial\Omega_N} v\hat{t} \, d\gamma \quad \forall v \in H^1_0
\]

(2)

3. FEM (Galerkin’s method with solution approximation by shape functions)

- Domain (with heat source \( f = 60 \) for \( x > 1 \), all quantities are dimensionless) and its discretization with 2 finite elements (4 nodes)

- Selected shape functions - \( \varphi_1(x, y) \), \( \varphi_2(x, y) \)

- Continuous approximation of the solution

\[
  u_h(x, y) = \alpha_1 \varphi_1(x, y) + \alpha_2 \varphi_2(x, y) + \alpha_3 \varphi_3(x, y) + \alpha_4 \varphi_4(x, y)
\]

\( \alpha_1, \alpha_2, \ldots, \alpha_N \) - unknown parameters - degrees of freedom (d.o.f.)
- **Galerkin’s method**
  
  \[ v \in \{ \varphi_1, \varphi_2, \varphi_3, \varphi_4 \} \Rightarrow \]
  
  4 algebraic equations - "virtual work" for "virtual" displacements \( \varphi_1, \varphi_2, \varphi_3, \varphi_4 \)
  
  Let’s assume the following notations:
  
  \[
  (\varphi_i, \varphi_j)_m = \alpha_j \int_{e_m} k \nabla \varphi_i \cdot \nabla \varphi_j \, dx \, dy \\
  (\varphi_i)_m = \int_{e_m} \varphi_i f \, dx \, dy + \int_{\partial \Omega_N \cap \partial e_m} \varphi_i \hat{n} \, ds
  \]
  
  \[
  \begin{align*}
  (\varphi_1, \varphi_1)_1 & + (\varphi_1, \varphi_2)_2 + (\varphi_1, \varphi_3)_3 + (\varphi_1, \varphi_4)_4 & = & & (\varphi_1)_1 + (\varphi_1)_2 \\
  (\varphi_2, \varphi_1)_1 & + (\varphi_2, \varphi_2)_2 + (\varphi_2, \varphi_3)_3 + (\varphi_2, \varphi_4)_4 & = & & (\varphi_2)_1 + (\varphi_2)_2 \\
  (\varphi_3, \varphi_1)_1 & + (\varphi_3, \varphi_2)_2 + (\varphi_3, \varphi_3)_3 + (\varphi_3, \varphi_4)_4 & = & & (\varphi_3)_1 + (\varphi_3)_2 \\
  (\varphi_4, \varphi_1)_1 & + (\varphi_4, \varphi_2)_2 + (\varphi_4, \varphi_3)_3 + (\varphi_4, \varphi_4)_4 & = & & (\varphi_4)_1 + (\varphi_4)_2 
  \end{align*}
  \]
  
  Entries in gray are equal to 0. Entries in red and green are integrals over elements 1 and 2 respectively.

- **m-th element (stiffness) matrix and (load) vector**
  
  \[
  K_{ij}^m = \int_{e_m} k \nabla \varphi_i \cdot \nabla \varphi_j \, dx \, dy, \quad f_i^m = \int_{e_m} \varphi_i f \, dx \, dy + \int_{\partial \Omega_N \cap \partial e_m} \varphi_i \hat{n} \, ds
  \]

- After assembling (element by element) one obtains global system of \( 4 \times 4 \) algebraic linear equations \( K \mathbf{u} = \mathbf{f} \).

- Accounting for Dirichlet (essential, kinematic) boundary conditions
  
  e.g. if \( \hat{u} = 3x - 2 \) is given at segment AD it implies that \( \alpha_1 = -2, \alpha_2 = 1 \) 
  
  Therefore, equations 1 and 2 are not needed.

- Postprocessing (performed element by element, with possible assessment of the result quality).

  For element 1: d.o.f. \( (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \) are already known, thus one may evaluate in this element
  
  approximation of solution: \( u_h(x, y) = \alpha_1 \varphi_1(x, y) + \alpha_2 \varphi_2(x, y) + \alpha_3 \varphi_3(x, y) \)
  
  and flux approximation: \( q_h = -k[\alpha_1 \nabla \varphi_1(x, y) + \alpha_2 \nabla \varphi_2(x, y) + \alpha_3 \nabla \varphi_3(x, y)] \)
  
  In general, \( u_h \) is continuous but \( q_h \) is not and the accuracy of \( u_h \) is worse than \( u_h \) accuracy. 
  
  ”Reaction” (normal component of flux) at the AD edge: \( q_n = q_h \cdot \mathbf{n} \)
(1) \[ K_{Q1} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \]

(2) \[ K_{Q2} = \frac{1}{2} \begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \]

4) Example (kind of b.o.) + Solution of the SALE

\[
\begin{bmatrix}
1 & 2 \\
-1 & -2 \\
0 & 0 \\
0 & 2
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4
\end{bmatrix}
= 
\begin{bmatrix}
5x - 2 \\
2x + y_1 + y_2 \\
2x + y_3 \\
2x + y_4
\end{bmatrix}
\]

\[
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4
\end{bmatrix}
= 
\begin{bmatrix}
5x \\
1
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
u_1 = 5x \\
u_2 = -2 \\
u_3 = 5 \\
u_4 = 2 \end{bmatrix}
\]
4. Homework

Consider problem (1) in the domain shown in this section (item 3).

- Write the formulas for FEM approximation of the solution, its flux and $q_n$ along the AD segment, knowing that $\alpha_3 = 2$, $\alpha_4 = -2/3$.
- Prove that such an approximation is continuous and satisfies the Dirichlet boundary condition. Evaluate $\alpha_1, \alpha_2$ for $\hat{u} = x^2$
- Check accuracy of satisfying the Neumann b.c. if $\hat{t}_1 = -1$, $\hat{t}_2 = 0$, $\hat{t}_3 = 1 - x$
3 Higher order finite element approximation

1. An exemplary problem - 1D bar (axial deformations)

\[ q(x,t) - \text{continuous} \]

Newton's principle: \( \frac{d}{dt}p = F \quad \forall \omega \subset \Omega, t \in [0,T] \)

Hamilton's principle: \[ \delta \int_0^T (K - U + W) \, dt = 0 \]
K - kinetic energy, U - potential energy, W - work done by loading

2. Strong formulation: \textit{Find} \( u(x,t) \in C^2 \), \textit{such that}

\[
\begin{align*}
Ap\dddot{u} + Ac\ddot{u} - AEu'' &= q(x,t) \quad \forall x \in (0,l), \forall t \in [0,T] \\
u(0,t) &= 0 \quad \forall t \in [0,T] \\
AEu'(l,t)n(l) &= P(t) \quad \forall t \in [0,T] \\
u(x,0) &= u_0(x) \quad \forall x \in [0,l] \\
\dot{u}(x,0) &= v_0(x) \quad \forall x \in [0,l]
\end{align*}
\] (4)

3. Weak formulation: \textit{Find} \( u(x,t) \in H_1 \), \textit{such that} \( u(0,t) = 0 \ \forall t \) \textit{and}

\[
\int_0^l vA\ddot{u} \, dx + \int_0^l vAc \ddot{u} \, dx + \int_0^l v'AEu' \, dx = \int_0^l vq \, dx + v(l)P \quad \forall v \in V_0, \forall t \in [0,T] \] (5)
4. FEM (Galerkin’s method and solution approximation by shape functions)

Let’s assume that $\dot{u} = 0 \forall t$. Then $u = u(x)$. 
• Discretization (finite elements)

• **Shape functions** (e.g. on element 3, using local enumeration)
  element by element algorithm

\[
\hat{\phi}_1(x) = \frac{(x - x_2)}{(x_1 - x_2)}, \quad \hat{\phi}_2(x) = \frac{(x - x_1)}{(x_2 - x_1)}, \quad \hat{\phi}_3(x) = (x - x_1)(x - x_2)
\]

remes: \( \hat{\phi}_1(x) + \hat{\phi}_2(x) = 1, \quad \hat{\phi}_1(x) + \hat{\phi}_2(x) + \hat{\phi}_3(x) \neq 1 \)

position of the third node is neither specified nor used

• Approximation of a solution (over element 3)

\[
u_h(x) = \hat{\alpha}_1 \hat{\phi}_1(x) + \hat{\alpha}_2 \hat{\phi}_2(x) + \hat{\alpha}_3 \hat{\phi}_3(x), \text{ d.o.f.} \quad \hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3 - \text{degrees of freedom (d.o.f.)}
\]

\[
\hat{\alpha}_1 = u_h(\hat{x}_1), \quad \hat{\alpha}_2 = u_h(\hat{x}_2)
\]

• Approximation of the geometry (on element 3)
\[ x = \hat{x}_1 \hat{\varphi}_1(x) + \hat{x}_2 \hat{\varphi}_2(x) \]

- **Galerkin's method**

\[ \int_0^l \varphi'_i A E u'_h \, dx = \int_0^l \varphi_i q \, dx + \varphi_i(l) P \quad \forall \varphi_i, \ i = 1, \ldots, N \]  \(6\)

- **Element stiffness matrix and load vector**

\[ K_{ij}^e = \int_e \hat{\varphi}'_i A E \hat{\varphi}'_j \, dx, \quad P_i = \int_e \hat{\varphi}_i q \, dx \]

or in a matrix form (for a 2 dof element)

\[ K^e = \int_e B^T D B \, dx, \quad B = [\hat{\varphi}'_1 \hat{\varphi}'_2 \hat{\varphi}'_3], \quad D = A E \]

\[ P^e = \int_e N^T q \, dx, \quad N = [\hat{\varphi}_1 \hat{\varphi}_2 \hat{\varphi}_3] \]

- **Assembling**

- **Accounting for kinematic conditions**
- **SLE solution; \( K u = P (+F) \); \( F \) - nodal forces for bar structures only**
- **Postprocessing with assesment of the result quality**

  e.g. for element 3, d.o.f. \( (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3) \) are already known, thus

  continuous approximation of axial displacement: \( u_h(x) = \hat{\alpha}_1 \hat{\varphi}_1(x) + \hat{\alpha}_2 \hat{\varphi}_2(x) + \hat{\alpha}_3 \hat{\varphi}_3(x) \)

  discontinuous approximation of axial stress: \( \sigma_h(x) = E \frac{du_h}{dx} = \hat{\alpha}_1 \hat{\varphi}'_1(x) + \hat{\alpha}_2 \hat{\varphi}'_2(x) + \hat{\alpha}_3 \hat{\varphi}'_3(x) \)

  discontinuous approximation of axial force: \( S_h(x) = A \sigma_h(x) \)

  improved (in fact exact) axial nodal values of axial forces: \( F^e = K^e u^e - P^e \)

- **Homework**

  Consider the following 1D bar problem

  Find \( u(x) \in H_1([a,b]) \), such that \( u(a) = u(b) = 0 \) and

\[ \int_a^b v' A E u' \, dx = \int_a^b v q_0(l - x) \, dx \quad \forall v \in V_0, \ q_0 \in R, \ l = b - a \]  \(7\)
– Calculate the matrix and vector for the three node finite element [2,3] with hierarchical shape functions up to the second order.
– Assemble the matrix and vector assuming that the global node numbers are 1,3,2 (left, mid, right).
– Write the formulas used for FEM approximation of axial displacement and axial force over that element.
– Discretize a bar represented by segment [1,3] with 2 second order hierarchical elements.
– Calculate FEM approximation of the axial force and the improved axial forces.
– Check the global equilibrium equation.

4 FEM analysis of mechanical vibrations

Vibrations - incredible common phenomenon. In certain cases they are a positive thing (speech, music) and sometimes a negative one (noise of braking pads, vibration of buildings, bridges,...). Engineers are involved in both making and suppressing vibrations.

1. Types of the related problems
   • Eigen vibrations.
   • Response to time dependent loading (periodic or random).
   • Wave propagation.

2. Single DOF system - without external excitation

\[
\begin{align*}
    &\begin{array}{c}
    \text{k} \\
    \text{c}
    \end{array}
    \begin{array}{c}
    \text{m} \\
    \text{x}
    \end{array}
    \\
    &\begin{array}{c}
    \delta \\
    \zeta
    \end{array}
    \begin{array}{c}
    \omega \\
    \omega_d
    \end{array}
    \\
    &\begin{array}{c}
    \alpha_1,2 = \omega (-\zeta \pm \sqrt{\zeta^2 - 1}) \in C, \\
    x = A_1 e^{\omega_1 t} + A_2 e^{\omega_2 t}, \\
    e^{i\theta} = \cos(\theta) + i \sin(\theta)
    \end{array}
\end{align*}
\]

\[
\begin{align*}
    &\text{δ} = \frac{c}{2m}, \quad \zeta = \frac{\delta}{\omega} - \text{damping ratio} \\
    &\alpha_{1,2} = \omega (-\zeta \pm \sqrt{\zeta^2 - 1}) \in C, \quad x = A_1 e^{\omega_1 t} + A_2 e^{\omega_2 t}, \quad e^{i\theta} = \cos(\theta) + i \sin(\theta)
\end{align*}
\]

\[
\begin{align*}
    &\text{• } \zeta = 0 \rightarrow m\ddot{x} + kx = 0 \rightarrow x = A \sin(\omega t - \varphi) \text{ or } x = A \sin(\omega t) + B \cos(\omega t) \\
    &\quad -mA_\omega^2 \cos(\omega t - \varphi) + kA \cos(\omega t - \varphi) = 0 \\
    &kA = \omega^2 mA, \quad A \neq 0, \quad \omega = \sqrt{\frac{k}{m}} - \text{natural (angular) frequency} \\
    &\text{initial conditions imply } A \text{ (amplitude) and } \varphi \text{ (phase), but typically they are not important}
\end{align*}
\]

\[
\begin{align*}
    &\text{• } 0 < \zeta \leq 1 \rightarrow x = A e^{-\zeta\omega t} \cos(\omega_d t - \varphi), \\
    &\omega_d = \omega \sqrt{1-\zeta^2} - \text{damped natural 'frequency'} \\
    &\text{experimental estimation of damping ratio: } \zeta \approx \frac{0.11}{m50%}
\end{align*}
\]
1. \( 1 < \zeta \rightarrow x = A_1 e^{\alpha_1 \omega t} + A_2 e^{\alpha_2 \omega t}, \) 
\( \alpha_1, \alpha_2, \omega \in \mathbb{R} \)

3. Axial vibrations of elastic bars

![Wave propagation diagram](image1)

Particles vibrate in the direction of wave propagation.

4. Axial free vibrations of elastic bars

Standing (stationary, fixed in space) waves - combination (interference) of two waves moving in opposite directions, each having the same amplitude and frequency. When waves are superimposed, their energies are either added together or cancelled out. The PDE of motion reduces in this case to the following problem.

\[
\begin{align*}
\rho \ddot{u} - Eu'' &= 0 \quad \forall x \in (0, l), \forall t \in [0, \tau] \quad \text{wave (string) equation} \\
u(0, t) &= 0 \quad \forall t \in [0, \tau] \\
u'(l, t) &= 0 \quad \forall t \in [0, \tau]
\end{align*}
\]

Separation of variables, \( u(x, t) = U(x)V(t) \) implies \( \frac{V''}{V} = \frac{\ddot{U}}{U} = \text{const} \) and results in two eigenproblems (the Helmholtz equations)

\[
\begin{align*}
\ddot{V} + \omega_n^2 V &= 0 \\
U'' + k_n^2 U &= 0
\end{align*}
\]

where \( k_n = \omega_n \sqrt{\frac{E}{\rho}} \) are wave numbers and the general, nontrivial solution is of the form

\[
u(x, t) = \sum_{n=1}^{\infty} \left[ A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \right] \left[ (C_n \cos(k_n x) + D_n \sin(k_n x)) \right],
\]

\[
\omega_n = 2\pi f_n, \quad \omega_n = \frac{2\pi}{T}, \quad k_n = \frac{2\pi}{\lambda_n}, \quad \frac{\lambda_n}{Tn} = \frac{k_n}{\omega_n} = \sqrt{\frac{E}{\rho}} \quad - \text{sound (phase) speed}
\]

The Dirichlet b.c. implies that \( C_n = 0 \)

The Neumann b.c. implies the characteristic equation

\[
cos(k_n l) = 0 \quad \Rightarrow \quad k_n l = (2n - 1)\frac{\pi}{2}, \quad n = 1, 2, 3, \ldots
\]
and one obtains the following angular frequency spectrum and mode shapes

\[ \omega_n = (2n - 1) \frac{\pi}{2l} \sqrt{\frac{E}{\rho}}, \quad u_n(x) = \beta \sin[(2n - 1) \frac{\pi}{2l} x] \]  

5. Finite element analysis of free vibrations

- Formulation (see Eq. (61) for 1D axially loaded bar)
- Approximation: \( u_h(x, t) = \alpha_1(t)\varphi_1(x) + \alpha_2(t)\varphi_2(x) + \ldots + \alpha_N(t)\varphi_N(x) \)
- In general, there are 3 element matrices
  - mass matrix \( M^{(e)} = \int_e N^T A \rho N \, dx \)
  - damping matrix \( C^{(e)} = \int_e N^T A c N \, dx \)
  - stiffness matrix \( K^{(e)} = \int_e B^T A E B \, dx \)
- After assembling these matrices and accounting for essential b.c. one obtains the following system of second order ODEs

\[
\begin{cases}
\bar{M}\ddot{u} + \bar{C}\dot{u} + \bar{K}u = \bar{f}(t) & \forall t \in (0, T) \\
u(t = 0) = u_0 \\
\dot{u}(t = 0) = v_0 
\end{cases}
\]

- Free undamped vibrations \( \Rightarrow \) eigenproblem
  Find \( u \neq 0 \) such that \( \bar{M}\ddot{u} + \bar{K}u = 0 \ \forall t \in (0, T) \)

Since solution has the following form
\[ u = U \sin(\omega t + \psi) \]

one obtains
\[ (-\omega^2 \bar{M} U + \bar{K} U) \sin(\omega t + \psi) = 0 \quad \forall t \in (0, T) \]

that reduces to the following generalized algebraic eigenproblem

\[
\begin{cases}
    KU = \omega^2 \bar{M} U \\
    U^T \bar{M} U = \mu_0,
\end{cases}
\]

\[ \mu_0 \text{ is an arbitrary constant, e.g. } 1 \text{ kg} \quad (13) \]

which, in turn, enables computation of approximate mode shapes \((U_1, U_2, \ldots)\) and corresponding natural frequencies \((\omega_1, \omega_2, \ldots)\)

- Due to the symmetry of \(\bar{M}\) and \(\bar{K}\) the eigen modes are orthogonal, i.e.

\[
\begin{align*}
    U_i^T \bar{M} U_j &= 0 \quad \text{for } i \neq j \\
    U_i^T \bar{K} U_j &= 0 \quad \text{for } i \neq j \\
    U_k^T \bar{M} U_k &= 1 \\
    U_k^T \bar{K} U_k &= \omega_k^2
\end{align*}
\]

for normalized \(U_k\)

- example
Osiowe drgania plata

\[ l = 0.75 \mu m \]
\[ A = 25 \cdot 10^{-4} \text{m}^2 \]
\[ E = 207 \text{GPa} = 207 \cdot 10^9 \text{Pa} \]
\[ g = 7.800 \text{kg}/\text{m}^3 \]
\[ \Rightarrow c = \frac{E}{\sqrt{\rho}} = 5.15 \text{km/s} \]
(produkcja drutów)

* dyskretyzacja

\[ N=4 \quad h = \frac{h}{N} = 0.25 \]

* agregacja

\[ V^e = \frac{AE}{h} \begin{bmatrix} -1 & 1 \\ 4 & -4 \\ 1 & 0 \end{bmatrix}, \quad V^e = \frac{Agh}{6} \begin{bmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \]

\[ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \frac{c^2}{2Gc^2} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \]

* war. kinematyczny \( U_n = 0 \) → zerowanie + ma przelączkę
allo "krótka" \((4 \times 4 \rightarrow 3 \times 3)"

rozwiązanie uogólnionego problema nieskończonego

\[ \text{det} \begin{bmatrix} -4 \lambda & -\lambda & 0 \\ -\lambda & 2-4 \lambda & -\lambda \\ 0 & -\lambda & 2-4 \lambda \end{bmatrix} = 0 \]
\[ \lambda = \frac{c^2}{2Gc^2} \]
\[ \lambda > 0 \]

numerycznie: faktyczna \( V = L L^T + \text{Jacobian}(r) \)

* post processing \( \lambda_1 = 0.04675 \), \( \lambda_2 = 0.5 \), \( \lambda_3 = 1.646 \)

spectrum coelaboracji \( f = \frac{c}{2\pi} \sqrt{\frac{\lambda_i}{\rho}} \)
\( f = [1737, 5680, 10305] \text{ Hz} \)
\( [974, 5152, 8566] \)

\( \text{Prz. 1%, 10%, 10%} \)
6. **Numerical integration in time**

For the initial value problem \( \frac{dy}{dt} = f, \ y(t_0) = y_0 \)

one introduces discretization in time represented by time step \( \Delta t \).
Thus, \( y(t_0 + \Delta t) = y_0 + \int_{t_0}^{t_0 + \Delta t} f \, dt \)

The last integral is evaluated after assuming an approximation for integrand \( f \), that is equivalent to assuming certain approximation of \( y \) with respect to time.

7. Types of numerical methods for integration in time
   - Explicit and implicit.
   - Conditionally or unconditionally stable.

8. Newmark’s method (for \( \beta = 0.5, \gamma = 1 \))
   Without damping \( \ddot{M} \ddot{u} + \dot{K} \dot{u} = f(t) \ \forall t \in (0, T) \)
   - For given: \( u_0, \ v_0, \ \ddot{M}a_0 = f_0 - K \dot{u}_0 \)
   - \( (\ddot{M} + \frac{1}{2} \Delta t^2 \dot{K})a_{k+1} = f_{k+1} - K(u_k + \Delta t \dot{v}_k) \)
   - \( u_{k+1} = u_k + \Delta t \dot{v}_k + \frac{1}{2} \Delta t^2 a_{k+1} \)
   - \( v_{k+1} = v_k + \Delta t a_{k+1} \)

9. Modal analysis (a good approximation for harmonic response)
   - \( u = z_1(t)U_1 + z_2(t)U_2 + \ldots + z_N(t)U_N, \ \text{ } U_i - \text{eigen vectors} \)
   - \( \ddot{z}_i + \zeta_i \dot{z}_i + \omega_i^2 z_i = F_i, \ \zeta_i = \alpha + \beta \omega_i^2 \ \forall i = 1, ..., N \) (decoupled system of ODEs)

10. Fourier analysis
   - Fourier series for a periodic function \( (T = 2l) \):
     \[
     FS(f)(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n \pi x}{l} + b_n \sin \frac{n \pi x}{l} \right) = \sum_{n=-\infty}^{\infty} C_n e^{in \pi x / l}
     \]
     where
     \[
     a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n \pi x}{l} \, dx, \quad a_0 = \frac{1}{2l} \int_{-l}^{l} f(x) \, dx
     \]
     \[
     b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n \pi x}{l} \, dx, \quad C_n = \frac{1}{2l} \int_{-l}^{l} f(x) e^{-in \pi x / l} \, dx
     \]
   - \( \frac{n \pi}{l} = \omega_n, \ \text{ } f_n = \frac{n}{T} \) - sequence of angular frequencies (infinite spectrum)
A function may be represented both in time and frequency domains

**TIME domain** - \( f(x) \)

**FREQUENCY domain** - \(|C_n| (\omega_n = \frac{n\pi}{T})\)

\[A_i = \sqrt{a_i^2 + b_i^2}\]

For non-periodic function \((T = \infty)\)

Fourier transform to frequency domain

\[a(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos(\omega x) \, dx, \quad b(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin(\omega x) \, dx, \quad C(\omega) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \omega x} \, dx\]

Inverse transform

\[f(x) = \int_{-\infty}^{\infty} [a \cos(\omega x) + b \sin(\omega x)] \, d\omega = \int_{-\infty}^{\infty} C(\omega) e^{2\pi i \omega x} \, d\omega\]

**TIME domain** - \( f(x) \)

**FREQUENCY domain** - \( C(\omega) \) \((\omega \in R)\)

In practice: DFT, FFT
11. Homework

- Use 2 finite elements with linear shape functions for a bar and
  - compute global stiffness and mass matrices for such a discretization
  - calculate spectrum of natural frequencies and draw the corresponding mode shapes
  - verify orthogonality of the modes, calculate modal masses and stiffnesses
  - repeat the above proposed analysis for a lumped mass matrix
  - Explain differences between single and multi step methods, explicit and implicit schemas, stable and unstable methods.
  - Sketch graphs of \( f(t) = 2 \cos(\pi t) - \sin\left(\frac{\pi t}{2}\right) \) in time and frequency domains.

- Knowing that \( u(x,t) = \sum_{n=1}^{\infty} [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)] [(C_n \cos(k_n x) + D_n \sin(k_n x)] \) is a general form of the eigen function (exact free vibration displacements) for a bar clamped at both ends, determine the bar eigen frequencies and eigen modes.

5 Linear elasticity

1. Principle of virtual work (primal weak formulation):

   Find continuous \( u(x,y,z) \in H^1(\Omega) + \hat{u} \), such that \( u = \hat{u} \) on \( \partial \Omega \) and

   \[
   \int_{\Omega} \varepsilon(v) : \sigma(u) \, dx \, dy \, dz = \int_{\Omega} v \cdot f \, dx \, dy \, dz + \int_{\partial \Omega} v \cdot t \, ds \quad \forall v \in H^1_0
   \]  

   \( \sigma = C \varepsilon \) (21 independent parameters reduce to 2 for isotropic materials)

   \( \sigma_{ij} = 2\mu \varepsilon_{ij} + \lambda \delta_{ij} \varepsilon_{kk} \)

   \( \alpha \varepsilon_{ij} \varepsilon_{ij} \leq C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \leq \beta \varepsilon_{ij} \varepsilon_{ij} \), \( \alpha, \beta > 0 \) (positive definitness and ellipticity)

2. Plane strain state (3D \( \rightarrow \) 2D), \( \varepsilon_{33} = 0 \), \( \sigma_{33} = \frac{1}{2(\mu+\lambda)}(\sigma_{11} + \sigma_{22}) \)

Figure 1: Plane strain example. An infinite long body, its section of arbitrary length \( d \) and selected cross section \( S \) used for representing the whole body.

Find continuous \( u(x,y) \in H^1(\Omega) + \hat{u} \), such that \( u = \hat{u} \) on \( \partial \Omega \) and
$$\int_{S} \varepsilon(v) : \sigma(u) \, dx \, dy = \int_{S} v \cdot f \, dx \, dy + \int_{\partial S_{t}} v \cdot \hat{t} \, ds \quad \forall v \in H_{0}^{1}$$  \hspace{1cm} (15)$$

Even if the assumptions of the plane strain state are not met exactly one may use this simplification of a 3D state being aware of introducing an additional modeling error.

3. Plane stress state (3D $\rightarrow$ 2D),  $\sigma_{33} = 0, \quad \varepsilon_{33} = -\frac{\lambda}{2\mu+\lambda}(\varepsilon_{11} + \varepsilon_{22})$

Figure 2: Plane stress example - a plate with non constant thickness $h(x, y) = z_{2} - z_{1}$, modeled by equations formulated in the S domain ($z=0$)

$$\text{Find continuous } u(x, y) \in H^{1}(\Omega) + \hat{u}, \text{ such that } u = \hat{u} \text{ on } \partial\Omega_u \text{ and}$$

$$\int_{S} \varepsilon(v) : \sigma(u) h \, dx \, dy = \int_{S} v \cdot f h \, dx \, dy + \int_{\partial S_{t}} v \cdot \hat{t} h \, ds \quad \forall v \in H_{0}^{1}$$  \hspace{1cm} (16)$$

$$h = h(x, y)$$

$$-\text{div}(h\sigma(u)) = h\dot{f} \quad \text{in } S$$

$$\sigma\dot{u} = \dot{t} \quad \text{on } S_{t}$$

$$u = \hat{u} \quad \text{on } S_{u}$$  \hspace{1cm} (17)$$

The plane stress state is a simplified model of 3D solid body associated with extension of a free end condition that dominates if the thickness of the part is small enough.

4. Plane strain vs. plane stress states

For an isotropic material a plane strain state solution may be obtained by finding solution to an auxiliary plane stress problem with arbitrary constant thickness $h$ and modified material parameters in the following way

$$E \leftarrow \frac{E}{1-\nu^{2}} \quad \nu \leftarrow \frac{\nu}{1-\nu}$$  \hspace{1cm} (18)$$

Such a conversion makes sense only for $\nu \leq 1/3$ since for larger Poisson ratios the modified values are greater than 0.5 and the auxiliary problem has no solution.
5. Fundamental solution in 2D

\[ G_{ij} = \frac{1}{8\pi\mu(1-\nu)} \left[ (3 - 4\nu) \ln r + r_i r_j \right] \] (19)

6. **Do not use pointwise kinematic boundary conditions in 2D and 3D for problems of second order!**

7. **Pointwise loading for these problems is also not reasonable!**

8. Solution singularities - an example

![Figure 3: Equivalent stress for a plane strain problem. Problem scheme and the Mises equivalent stress distribution with singularity at the reentrant corner and end points of the fixed support](image)

9. **Locking example**

6 **Coupled problem**

1. Formulation of a steady-state thermo-mechanical problem (with neglected interior source terms)

   Find \( u \in H^1_0(\Omega) + h \) and \( \theta \in H^1_0 + T \), such that:

   \[
   \begin{align*}
   \int_\Omega \varepsilon(v) : C \varepsilon(u) \, d\Omega - \int_\Omega \text{tr}(\varepsilon(v)) \alpha \theta \, d\Omega &= \int_{\partial\Omega} v q \, ds \\
   \int_\Omega \nabla \psi k \nabla \theta \, d\Omega &= \int_{\partial\Omega} \psi S \, ds
   \end{align*}
   \]

   (20)
Figure 4: Convergence (in logarithmic scales) of the true displacement error (left) and displacement norm (right) for rectangular plate of aspect ratio (h:L) 1:10 and 1:100. About 40 000 or 3 000 000 dof were needed for accuracy less than 0.1%. For less than 30 000 dof the elongated plate solution converges very slowly (about 10 times slower than expected).

∀ v ∈ H₀¹(Ω),  ∀ ψ ∈ H₀¹(Ω)
2. Element stiffness matrix and load vector \( \mathbf{N}_1 = [\varphi_1, 0, 0], \mathbf{N}_2 = [0, \varphi_1, 0], \ldots \)

\[
K_{ij}^{uu} = \int_e \varepsilon(N_i) : C \varepsilon(N_j) \, d\Omega \quad K_{ij}^{u\theta} = \int_e \text{tr}(\varepsilon(N_i)) \alpha \varphi_j \, d\Omega \quad (21)
\]

\[
K_{ij}^{\theta\theta} = \int_e \nabla \varphi_i \cdot \nabla \varphi_j \, d\Omega \quad (22)
\]

\[
P_{i}^u = \int_{\partial \Omega_t} N_i q \, ds \quad (23)
\]

\[
P_{i}^\theta = \int_{\partial \Omega_s} \varphi_i S \, ds \quad (24)
\]

3. Assembling element matrices (vectors) into global matrix (vector) - on the basis of d.o.f. connectivities (local with global numbering relations)

4. Exemplary 2D domain and discretization by 3 elements

5. Selected scalar \( \varphi_i \) vertex and edge shape functions
6. Static boundary conditions (only distributed loading is used in well posed problems)
Non-zero entries of load vector (using local node numbering) for the triangular element

\[
\hat{P}_4^u = \int_{\partial e \cap \Omega_t} [0, x/2][0, q(x)]^T \, ds 
\]

\[
\hat{P}_6^u = \int_{\partial e \cap \Omega_t} [0, x(1 - x/2)][0, q(x)]^T \, ds 
\]

\[
\hat{P}_8^u = \int_{\partial e \cap \Omega_t} [0, 1 - x/2][0, q(x)]^T \, ds 
\]

7. Kinematic boundary conditions
Both components of displacement along the part of the boundary \((x = 0, 1 \leq y \leq 3)\) are 0.
Never use pointwise kinematic boundary conditions in 2D and 3D problems of second order!

8. Postprocessing
\[
u = \sum \alpha_k N_k \text{ (continuous)} \Rightarrow \varepsilon, \sigma \text{ (discontinuous)} 
\]

9. A posteriori error estimation and mesh adaptation

Figure 5: Heterogeneous material distribution (colors represent different materials) and \(hp\)-adapted FEM mesh (colors represent order of approximation). Note the "hanging".
10. **Homework**

- For the plane strain problem and discretization by 3 elements
  - sketch graphs of all shape functions
  - calculate load vectors for all elements
  - calculate displacements at one interior point of each element for the following d.o.f.
    \[ [0, 0, 0, 0, 0, 0.5, -1, 1, -2, 1, -2, 2, -3, 2, -2.5] \times 10^{-2} \]

11. Mixed (the Hellinger-Reissner principle), 2 field

- good coarse mesh accuracy for stresses
- no problems with the incompressible material \((\nu = 0.5)\)
- no sensitivity against mesh distortions
- no sensitivity against heterogeneous materials with significantly different material properties
- ...

Find \( \sigma \in H(\text{div}, \Omega, S) \), \( \sigma n = \hat{t} \) on \( \partial \Omega_t \) and \( u \in L^2(\Omega, V) \):

\[
\begin{cases}
\int_{\Omega} \tau : C^{-1} \sigma \, d\Omega + \int_{\Omega} \text{div} \tau \cdot u \, d\Omega = \int_{\partial \Omega_t} \tau n \cdot \hat{u} \, ds & \quad \tau \in H(\text{div}, \Omega, S), \quad \tau n = 0_{\partial \Omega_t} \\
\int_{\Omega} v \cdot \text{div} \sigma \, d\Omega = -\int_{\Omega} v \cdot f \, d\Omega & \quad v \in L^2(\Omega, V)
\end{cases}
\]

(29)

Approximation of \( u \) must be continuous and of \( \sigma \) such that traction is continuous.
\[ \phi_i(x) = \phi_i \left[ \xi_i(v) \right] \]
\[ \varphi = \phi \cdot g \cdot \varphi \]
\[ g: x = a \phi_1(v) + b \phi_2(v) = a(x - x) + b \phi \]

**Zastosowanie**

\[ I = \int_a^b \frac{d\phi_1}{dx} \frac{d\phi_2}{dx} dx = ? \]

- **Rozwiązywanie**
  \[ \frac{d\phi_1}{dx} = \frac{d\phi_1}{dv} \frac{dv}{dx} \]
  \[ \frac{d\phi_2}{dx} = \frac{d\phi_2}{dv} \frac{dv}{dx} = (1 - 2v) \cdot \frac{4}{h} \]

- **Ostatecznie**
  \[ \int f(x)dx = \int \frac{1}{4} [x(v)] \frac{dv}{dx} dx \]
  \[ \int \frac{d\phi_1}{dx} \frac{d\phi_2}{dx} dx = \int \left( \frac{3}{4} \right) \cdot \frac{1 - 2v}{h} \cdot dv \]
  \[ = \frac{3}{4} \int (\phi_1 - \phi_2) dv \]

(postprocessing)

---

25
Współrzędne trójkątne (powierzchniowe)

\[ \begin{align*}
\Phi(x, y) &= \Phi_i \left\{ x \in [\lambda_1(x, y), \lambda_2(x, y), \lambda_3(x, y)], \frac{1}{2} \right\} \\
\lambda_1 &= \frac{A_1}{A}, \quad \lambda_1 + \lambda_2 + \lambda_3 = 1
\end{align*} \]

\[ \begin{align*}
\phi_i &= \frac{(y - y_3)(x - x_3)A_i}{(x - x_3)(y - y_3)} \\
\phi_i &= \frac{\lambda_2 \cdot \lambda_3 (\lambda_2 - y_3)}{y_3 \cdot y_3 \cdot y_3 - 4y_3} \\
\phi_{100} &= \frac{\lambda_1 \cdot \lambda_2 \cdot \lambda_3}{y_3 \cdot y_3 \cdot y_3}
\end{align*} \]
7 Discretization convergence - mathematical basis of FEM

Typically, only an approximate solution can be obtained and it is searched in a trial shape function space \( U_h = \text{span}(g_1, g_2, \ldots, g_n) \subset U \), Thus,

\[
u_h = \sum \alpha_i g_i \in U_h \subset U\tag{30}\]

Similarly, the test basis functions \( e_j, j = 1, 2, \ldots, n \) define a finite dimensional subspace \( V_h \subset V \).

The unknown d.o.f \((\alpha_1, \alpha_2, \ldots, \alpha_n)\) are computed by the Galerkin method

\[\egin{cases}
u_h \in U_h \\
b(u_h, v_h) = l(v_h) \
\forall v_h \in V_h
\end{cases}\tag{31}\]

If the discrete inf-sup condition

\[
\inf_{u_h \in U_h \setminus \{0\}} \sup_{v_h \in V_h \setminus \{0\}} \frac{|b(u_h, v_h)|}{|v_h||u_h|} = \gamma_h, \quad \gamma_h > 0
\tag{32}\]

holds, there exists a unique stable solution to the linear algebraic equations (31), i.e.

\[
||u_h||_U \leq \frac{1}{\gamma_h} ||l||_{V'}
\tag{33}\]

The \( \gamma_h \) constant depends on the assumed \( U_h \) and \( V_h \) spaces. If the \( \gamma_h \) constants admit a positive lower bound,

\[
\inf_h \gamma_h = \gamma_0 > 0
\tag{34}\]

i.e. a uniform discrete inf-sup condition holds, the following approximation error bound (convergence estimate) holds

\[
||u - u_h||_U \leq \frac{M}{\gamma_0} ||u - w_h||_U
\tag{35}\]

which states that convergence of the approximate solution follows from the stability (\( M/\gamma_0 \) constant value) and approximability (\( \inf_U ||u - w_h|| \)) of the discretization.

If condition (35) holds one may expect the following error decrease for nonsingular solution and uniform mesh refinements

\[
||e||_U \leq CN^{-p}
\tag{36}\]

where: \( p \) is the approximation order, \( N \) stands for the number of dofs, \( C \) is an unknown, problem dependent constant.

In general even if \( U_h \subset U \) as well as \( V_h \subset V \) and the continuous inf-sup condition holds the discrete inf-sup condition (37) may not be satisfied since the supremum on the left-hand side in the discrete condition is computed over a smaller space \( (V_h \subset V) \) than in the continuous one. Thus, the optimizer of the left-hand side of Eq. (37) may not belong to the assumed space \( V_h \), i.e.

\[
\sup_{v \in V \setminus \{0\}} \frac{|b(u, v)|}{||v||_V} \geq \gamma ||u||_U \Leftrightarrow \sup_{v_h \in V_h \setminus \{0\}} \frac{|b(u_h, v_h)|}{||v_h||_V} \geq \gamma_h ||u_h||_U
\tag{37}\]

It is worth mentioning that the coercivity of the \( b \) form, i.e fulfillment of the following inequality

\[
b(v, v) \geq \alpha ||v||^2 \quad \forall v \in V = U, \quad \alpha > 0
\tag{38}\]
implies the continuous inf-sup condition since

$$\sup_{v \in V \setminus \{0\}} \frac{|b(u, v)|}{||v||_V} \geq \frac{|b(u, u)|}{||u||_V} \geq \alpha ||u||_V \quad \Rightarrow \quad \gamma = \alpha$$

(39)

and also the discrete one. Indeed in such a case also $\gamma_h = \alpha$ since

$$\sup_{v_h \in V_h \setminus \{0\}} \frac{|b(u_h, v_h)|}{||v_h||_V} \geq \frac{|b(u_h, u_h)|}{||u_h||_V} \geq \alpha ||u_h||_V$$

(40)

Moreover, in such a case, FEM (the Galerkin approach) delivers the best possible approximation $u_h$ in the sense of the energy norm

$$||e|| = \sqrt{b(e, e)}$$

(41)

One may prove that for symmetric positive definite bilinear forms by error orthogonality and the Schwartz inequality

$$||u - u_h||^2 = b(u - u_h, u - u_h) = b(u - w_h + w_h - u_h, u - u_h) =$$

$$= b(u - w_h, u - u_h) + b(w_h - u_h, u - u_h) = b(u - w_h, u - u_h) \leq$$

$$\leq ||u - w_h|| ||u - u_h||$$

(42)

that implies

$$||u - u_h|| \leq ||u - w_h|| \quad \forall w_h \in V_h$$

(43)
Approximation error estimation

\[ u - y \] a function

\[ \| u \| - \text{measure of the error (length is a measure of a vector)} \]

\[ \| u \| : \mathbb{V} \rightarrow \mathbb{R}^+ \]

\[ \| u \| = 0 \quad \iff \quad u = 0 \]

\[ \| u \| = \| u \| \quad \text{for all } \| u \| \]

\[ \| u \| \leq \| u \| + \| u \| \]

\[ \| u \| = \int u^2 \, dx \]

\[ \| u \| = \sqrt{\int u^4 \, dx} \]

\[ \| u \| = \int (u')^2 \, dx \]

Example

\[
\begin{align*}
- u'' + u &= x^3 - 6x^2 + 12 & \forall x \in (0, 5) \\
\end{align*}
\]

\[ u(0) = 0 \]

\[ u(5) = 5 \]

Solution:

\[ u(x) = x^3 - 6x^2 + 6x \]

\[ \| u \|_0 = 12.7 \]

\[ \| u \|_2 = 16.0 \]
FEH analysis

\[ x = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \]

\[
\begin{array}{c|cccc}
X & 0 & 1 & 2 & 3 \\
0 & 0.238 & -0.80 & -0.68 & 5 \\
1 & -2 & 1 & 0 & 5 \\
2 & 0 & 1 & -7 & -8 & 5
\end{array}
\]

\[ \text{Relative error} \]

- \[ \frac{\|E\|_{\infty}}{\|E\|_{\infty}^0} = 13.2\% \]
- \[ \frac{\|E_1\|_{\infty}}{\|E_1\|_{\infty}^0} = 6.0\% \]

Contribution of elements to the error norm (localization)

\[ R_i = \frac{\|E_{\text{local}}^i\|_{\infty}}{\|E_{\text{local}}^i\|_{\infty} + \text{reference value}} \] (relative indicator)

<table>
<thead>
<tr>
<th>el (i)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>% [ e_0 ]</td>
<td>0.238</td>
<td>0.62</td>
<td>1.57</td>
<td>7.87</td>
</tr>
<tr>
<td>% [ e_1 ]</td>
<td>16.3</td>
<td>7.86</td>
<td>3.67</td>
<td>28.1</td>
</tr>
</tbody>
</table>
Hierarchical (Runge) error estimate

\[ e \approx \frac{u_{n-1} - u_n}{2} \]

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>4.5</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_n )</td>
<td>0</td>
<td>0.21</td>
<td>0.36</td>
<td>0.37</td>
<td>-1.8</td>
<td>-7.6</td>
<td>-23</td>
<td>-28</td>
<td>5</td>
</tr>
<tr>
<td>( u_{n-1} )</td>
<td>0</td>
<td>1.6</td>
<td>1</td>
<td>-1.2</td>
<td>-1.1</td>
<td>-5.3</td>
<td>-8.3</td>
<td>-25</td>
<td>5</td>
</tr>
</tbody>
</table>

![Graph showing error values and reference values]

\[ \| u_{n-1} - u_n \|_0 = 2.08 \]
\[ \| u_{n-1} - u_n \|_i = 6.37 \]

\[ \frac{\| e \|_i}{\| u_{n-1} \|_i} \approx 6.2\% \]
\[ \frac{\| e \|_i}{\| u_{n-1} \|_i} \approx 40.4\% \]

Error indicators:

\[ \| \frac{e_{i+1}}{u_{n-1}} \|_i = \delta_i \]

<table>
<thead>
<tr>
<th>( i )</th>
<th>1</th>
<th>2</th>
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<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta_i )</td>
<td>5.6</td>
<td>4.2</td>
<td>13</td>
<td>6.4</td>
</tr>
<tr>
<td>( \delta_i )</td>
<td>6.2</td>
<td>28</td>
<td>24</td>
<td></td>
</tr>
</tbody>
</table>
Residual error estimate

For the example \(- u'' + u = x^2 - 6x^2 + 12\)

\[ R(x) = x^2 - 6x^2 + 12 + u_n'' - u_n, \quad l = 0 \]

\[ R(x)\neq 0, \quad l(u_n) \]

Theorem

\[ \| \varepsilon \|_{\theta} \leq C \| R \|_{\theta} \]

\[ e = u - u_n \]

\[ \| \varepsilon \|_{\theta} \leq \| L(u_n + \varepsilon) - f \|_{\theta} = \| (L(u) = f - L(u_n)) \|_{\theta} \]

\[ \| R \|_{\theta} \leq 29.5 \]

\[ \| R \|_{\theta} \leq 46.5 \]

Elementwise

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2.2 & 8.5 & 47.0 & 337 & [\%]
\end{array}
\]
Zadanie
Zastosować metody residualna, wygładzenia oraz interpolacji do obliczenia wskaznika błędu aproksymacji MES rozwiązania równania: \(-u'' + 2u = 1\) w elemencie [2] z liniowymi funkcjami kształtu dla następujących wartości wezłowych:

<table>
<thead>
<tr>
<th>(x_i)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(u_i)</td>
<td>-1</td>
<td>0</td>
<td>2</td>
<td>-2</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
1) & \text{ dla katodo-głowego elementu:} \\
& u_n = u_1 \varphi_1(x) + u_2 \varphi_2(x) \\
& \Rightarrow u_n' = u_1 \varphi_1' + u_2 \varphi_2' \\
& (\frac{u_2 - u_1}{h}) \\
2) & \text{ interpolacja w węźle:} \\
& d_i = \frac{u_i - u_{i-1}}{h_i - h_{i-1}} \\
& \text{ dla } h_1 = h_2 = h \\
& s = \frac{d_1 + d_2}{2} \\
3) & \text{ uniforne podziale (dla katodo-głowego elementu):} \\
& u_n = u_1 \varphi_1 + u_2 \varphi_2 \\
& d_2 = -1 \\
& (\sum_{i=1}^{2} d_i^2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 1.5, \quad \gamma_2 = 0.3 \\
& Wartość pośrednia \quad \overline{\xi} = 1.3 \\
& \overline{\xi}_i = 1.8, \quad \overline{\xi}_2 = 1.4, \quad \overline{\xi}_3 = 1.7, \quad \overline{\xi}_4 = 1.3 \\
\end{align*}
\]
Interpolation error estimate $II$ (revisited)

\[ \| u - u_h \|_m \leq C \| u - u_h \|_{H^m} \quad \forall v_h \in V_h \]

\[ \| u - u_h \|_m \leq C h^{m-1} \| u \|_{H^m} \]

Here $p = 1$

\[ \| u - u_h \|_0 \leq C h^2 \| u \|_2 \quad \text{second Dirichlet} \]

\[ u_h = (\tilde{u}_h)' \]

\[ \int_0^1 \frac{\tilde{u}_h''(x)}{x} \, dx \]

\[ 2^m \int_0^1 [(\tilde{u}_h(x))_2^2 \, dx \]

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<th>1</th>
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<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_i^{(2)}$</td>
<td>6.7</td>
<td>2.2</td>
<td>9.3</td>
<td>10.9</td>
</tr>
</tbody>
</table>

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<thead>
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<td>2.2</td>
<td>9.3</td>
<td>10.9</td>
</tr>
</tbody>
</table>

$e = \text{absolute error}$ (not relative)

\[ e = [u - u_h, H^0, \Omega] \]

\[ e = 16 \]

\[ e = 16 \]

\[ e = 16 \]
# 9 Introduction to nonlinear analysis

1. **Linear problem assumptions**
   - Linear constitutive law (e.g., elasticity)
   - Infinitesimally small displacements
   - Small strains
   - Nature of b.c. remains unchanged
   - Gaps (debonding) do not appear during deformations

2. **Typical nonlinear problems**
   - Material nonlinearity
   - Large displacements, small strains
   - Large displacements, large strains
   - Change of b.c. during deformation (contact, free boundary)
   - Debonding of composite material components
3. Examples of applications and profits from nonlinear modeling

- Response of structures to extreme events
- Failures and deformations of soils
- Residual stress determination
- Structure life-time prediction
- Validation of linear models

4. MULTIPLE solutions or NO solution may exist

5. One dof example - rigid lever with rotary spring

**Linear model:**

\[ M_s = k_0 \theta, \quad M_P = Pl, \quad k_0 \theta - Pl = 0 \implies \theta = \frac{Pl}{k_0} \]

**Nonlinear model:**

\[ M_s = k_0 \tan \theta \text{ (constitutive law)} \]
\[ M_P = Pl \cos \theta \text{ (geometrical relation)} \]
\[ R = k_0 \tan \theta - Pl \cos \theta = 0 \implies \theta = ? \]

**The Newton-Raphson method**

Find \( \theta \) such that \( R(\theta) = 0 \), given \( \theta_0 \)

The Taylor formula: \( R(\theta_i + h) \approx R(\theta_i) + dR(\theta_i, h) = R(\theta_i) + hR'(\theta_i) \)

Thus \( R(\theta_i) + dR(\theta_i, h) = 0 \implies h, \theta_{i+1} = \theta_i + h \)

\[ R' = -Pl \sin \theta - \frac{k_0}{\cos^2 \theta} \]

For \( l = P = k_0 = 1, \theta_0 \approx \frac{Pl}{k_0} = 1 \)

\[ R_0 = -1.0171, \quad R'_0 = -4.2670, \quad h = -0.2384, \quad \theta_1 = 0.7616 \]
\[ R_1 = -0.2999, \quad R'_1 = -2.5994, \quad h = -0.0884, \quad \theta_2 = 0.6732 \]
\[ R_2 = -0.0157, \quad R'_2 = -2.2595, \quad h = -0.0069, \quad \theta_3 = 0.6663 \]
\[ R_3 = -0.0001, \quad R'_3 = -2.2362, \quad h = -0.0001, \quad \theta_4 = 0.6662 \]
6. 1D example - large strains

- Strong formulation

\[
\begin{align*}
-A \frac{d\sigma}{dx} &= q(x) \quad \forall x \in (0, l) \\
\sigma &= E \varepsilon \quad \forall x \in (0, l) \\
\varepsilon &= \frac{du}{dx} + \frac{1}{2} \left( \frac{du}{dx} \right)^2 \quad \forall x \in (0, l) \\
u(0) &= 0 \\
A\sigma(l)n(l) &= P
\end{align*}
\] (44)

- Weak formulation

Find \( u(x) \in H_1 \), such that \( u(0) = 0 \) and

\[
A \int_0^l v'\sigma(u) \, dx = \int_0^l vq \, dx + v(l)P, \quad A \int_0^l v'\sigma(u) \, dx = A \int_0^l v'E \left[ u' + \frac{1}{2} (u')^2 \right] \, dx
\]

or

\[
r(u) = A \int_0^l v'\sigma(u) \, dx - \int_0^l vq \, dx - v(l)P \quad \forall v \in V_0
\] (45)

- Newton-Raphson linearization

Given \( u_n \), find \( \psi(x) \in H_1 \), such that \( \psi(0) = 0 \) and

\[
r(u_n) + \delta r(u_n, \psi) = 0, \quad u_{n+1} = u_n + \psi
\] (47)

where for \( r = \int_a^b F(x, u, u') dx \), \( \delta r = \int_a^b \left( \frac{\partial F}{\partial u} \psi + \frac{\partial F}{\partial u'} \psi' \right) dx \), i.e.

\[
A \int_0^l v' \frac{d\sigma}{du'} |_{u=u_n} \psi' \, dx = \int_0^l vq \, dx + v(l)P - A \int_0^l v'\sigma(u_n) \, dx \quad \forall v \in V_0
\] (48)

then \( u_{n+1} = u_n + \psi \)
- Element tangent matrix and vector

\[
K_{ij}^e = A \int_{e} \varphi_i' \frac{d\sigma}{d\varphi_i} \varphi_j' \, dx,
\]

\[
f_i^e = \int_0^l \varphi_i q \, dx + \varphi_i(l) \sigma - A \int_0^l \varphi_i' \sigma(u_n') \, dx
\]

for linear shape functions and \( q = \text{const} \)

\[
K_{el} = \frac{AE}{h} (1 + u_x') \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad P_{el} = \frac{qh}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - AE \varepsilon h \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\]
• Results for $AE = 100$, $l = 1$, $q = 30$, $P = -10$

Figure 6: 20 finite elements. Displacement along the axis and $u(l)$ versus load level for nonlinear (in red) and linear (in green) models. Maksimum strain is equal to about 20%.

7. Elastic - plastic deformations

• Strong formulation in 1D ($t$ - pseudo time)

$$
\begin{cases}
-A \frac{d\sigma}{dx} = q(x) & \forall x \in (0, l), \forall t \\
\sigma = E(\varepsilon - \varepsilon^p) & \forall x \in (0, l), \forall t \\
\varepsilon = \frac{du}{dx} & \forall x \in (0, l), \forall t \\
\varepsilon^p = \varepsilon_0^p & \forall x \in (0, l), t = 0 \\
u(0) = 0 & \forall t \\
A\sigma(l)n(l) = P & \forall t
\end{cases}
$$

(51)

• Associated flow rule

$$\dot{\varepsilon}^p = \gamma \frac{\partial \Phi}{\partial \sigma}$$

(52)

$$\Phi(\sigma, \varepsilon^p) = |\sigma - H\varepsilon^p| - \sigma_Y \leq 0$$ - Huber-Mises-Hencky (von Mises) yield surface

$$\Phi \leq 0, \gamma \geq 0, \gamma \Phi = 0$$ - Kuhn-Tucker loading-unloading conditions

$$\dot{\varepsilon} = E_T \dot{\varepsilon} \quad E_T = E \quad or \quad E_T = \frac{E_H}{E+H}$$

• Weak formulation

Find $u(x) \in H_1$, such that $u(0) = 0$ and

$$A \int_0^l v'\sigma(u) \, dx = \int_0^l vq \, dx + v(l)P \quad \forall v \in V_0$$

(53)

• Newton-Raphson linearization

Given $u_k^n$, $\varepsilon^{p(n+1)} = \varepsilon^{p(n)}$, find $\psi(x) \in H_1$, such that $\psi(0) = 0$ and

$$A \int_0^l v' E_T(u_k^{(n+1)}) \psi' \, dx = \int_0^l vq \, dx + v(l)P - A \int_0^l v' \sigma_{k+1}^{(n+1)} \, dx \quad \forall v \in V_0$$

(54)

then $u_{k+1} = u_k + \psi$
Przestrzeń (wektorzystów), moduł nieprzestawny (rozw. indeksowy)

\[ \mathbf{e} = \mathbf{e} - \mathbf{e}^* \quad \Rightarrow \quad \mathbf{e} = \mathbf{e} - \mathbf{e}^* + \mathbf{e}^* \]

- powierzchnia przestrzeni (wektorów) 

\[ \Phi(e) = \|e\| - \frac{\mathbf{e} \cdot \mathbf{e}}{\|\mathbf{e}\|^2} \leq 0 \quad , \quad \|e\| = \sqrt{\mathbf{e} \cdot \mathbf{e}} \]

- wektor danych

\[ e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \quad , \quad \mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \quad , \quad \|e\| = \sqrt{\mathbf{e} \cdot \mathbf{e}} = \sqrt{e_1^2 + e_2^2 + e_3^2} \]

\[ \|e\| - \frac{\mathbf{e} \cdot \mathbf{e}}{\|\mathbf{e}\|^2} \leq 0 \quad \Rightarrow \quad |e| - \frac{\mathbf{e} \cdot \mathbf{e}}{\|\mathbf{e}\|^2} \leq 0 \]

- wznoszenie wektora

\[ \Phi(e) = \|e - \frac{\mathbf{e} \cdot \mathbf{e}}{\|\mathbf{e}\|^2} \mathbf{e}\| = \sqrt{e_1^2 - \frac{e_1^2}{\|\mathbf{e}\|^2} e_1^2 + e_2^2 - \frac{e_2^2}{\|\mathbf{e}\|^2} e_2^2 + e_3^2 - \frac{e_3^2}{\|\mathbf{e}\|^2} e_3^2} \]

\[ \frac{\mathbf{e} \cdot \mathbf{e}}{\|\mathbf{e}\|^2} \mathbf{e} = \frac{e_1^2}{\|\mathbf{e}\|^2} \mathbf{e} = e_1 \mathbf{e}_1 \]

\[ \Phi(e) = \|e - e_1 \mathbf{e}_1 - e_2 \mathbf{e}_2 - e_3 \mathbf{e}_3 \| = \sqrt{(e_1 - e_1 \mathbf{e}_1)^2 + (e_2 - e_2 \mathbf{e}_2)^2 + (e_3 - e_3 \mathbf{e}_3)^2} \]

\[ \|e\| - \|e - e_1 \mathbf{e}_1 - e_2 \mathbf{e}_2 - e_3 \mathbf{e}_3\| \leq 0 \]
Euler's theorem

\[ E = \text{Euler's theorem} \]

\[ \phi = \text{Euler's theorem} \]
Linearity:

\[ \sigma = \frac{1}{2} \sum_{i} \sigma_i - \sum_{i} \sigma_{i-1} \]

\[ \Delta \sigma = \int_{D} \sigma \, d\Omega - \int_{D} \sigma_{i-1} \, d\Omega \]

\[ B_{\mu} = \sum_{i} B_{\mu i} \]

\[ u \in \mathfrak{C} \]

\[ u(0) = 0 \]

\[ \int_{D} B_{\mu} E_{\mu i} \, d\Omega = \int_{D} \sigma_{i-1} \, d\Omega - \int_{D} \sigma_{i} \, d\Omega \]

\[ u_{\mu i} = \frac{1}{2} u_{\mu i} - \frac{1}{2} u_{\nu i} \]

Closest point projection (minimum norm)

Ward and Quenouille Gaussa

1. \[ B_{\mu i} = B_{\mu} + E \sigma \]

2. \[ \sigma(B_{\mu i}) > 0 \]

\[ \Delta Y = \frac{\sigma(B_{\mu i})}{E + H} \]

\[ \sigma(B_{\mu i}) = -\alpha \]

\[ \sigma(E) = E \alpha \]

\[ \mathcal{J}_{i} = \mathcal{J}_{i} + \delta \mathcal{J}_{i} \]

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Radial return for every Gauss point
\[ \sigma^{tr} = E(\varepsilon_n - \varepsilon_n^p + \Delta \varepsilon) \]
If \( \Phi(\sigma^{tr}) > 0 \) then \( \Delta \gamma = \frac{\Phi(\sigma^{tr})}{E + H} \)

Results for a composite bar (two materials)

![Composite bar diagram](image)

Figure 7: Displacement versus load. Comparison of analytical and numerical solutions.

**Homework**
- Perform one step of the Newton-Raphson method for problem (48).
- For a given \( \varepsilon^p \) in a bar compute displacements and stresses using 2 finite elements.

## 10 Selected other computer methods

1. **XFEM - extended FEM**
   - A FEM version for modeling discontinuities by enrichment of selected shape functions by the Heaviside function. This way, also theoretically predicted solution behavior is built into the approximation (e.g. in vicinity of 2D crack tip by \( \sqrt{r} \sin(\frac{\theta}{2}) \), \( \sqrt{r} \cos(\frac{\theta}{2}) \), ...).
   - Example of enrichment of selected shape function, let's say \( \varphi_I \)
     \[ u_h = \ldots + u_I \varphi_I + \ldots \Rightarrow u_h = \ldots + (u_I + a_I \Psi_a + b_I \Psi_b) \varphi_I + \ldots \] (55)
   where \( \Psi_a \varphi_I, \Psi_b \varphi_I \) are additional shape functions attributed to the same node as the \( \varphi_I \) function and \( a_I, b_I \) are the corresponding additional degrees of freedom at that node.
   - 1D example

2. **BEM - boundary element method**
   - Let’s consider the following 2D model problem
     \[ -k \Delta u = f(x, y) \quad \text{in} \quad \Omega \]
     \[ u = \hat{u} \quad \text{on} \quad \partial \Omega_u \]
     \[ ku = \hat{q} \quad \text{on} \quad \partial \Omega_q \] (56)
   - Somigliana’s identity
     \[ cu(\xi) = \int_{\partial \Omega} [u^*(x, \xi)t(x) - t^*(x, \xi)u(x)] d\gamma + \int_{\Omega} u^*(x, \xi)f(x) \, d\Omega \] (57)
   where
     \[ v^* = -\frac{1}{4\pi k} ln r^2, \quad t^* = -\frac{1}{2\pi} \frac{(x_1 - \xi_1)n_1 + (x_2 - \xi_2)n_2}{r^2}, \quad r^2 = (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 \] (58)
Figure 8: A 1D basis shape function ($\varphi_I$) and its two enrichments ($\Psi_a\varphi_I, \Psi_b\varphi_I$) by distance from $a=0.75$ and Heaviside function, i.e. $\Psi_{a} = |x - 0.75|$, $\Psi_{b} = H(x - 0.75)$.

- Boundary integral equation (for $u_{|\partial\Omega} = 0$)
  \[
  \int_{\partial\Omega} u^*(x, \xi) t(x) \, d\gamma + \int_{\Omega} u^*(x, \xi) f(x) \, d\Omega = 0 
  \]  
  (59)

- Discretization for $t \in L^2(\partial\Omega)$ by piecewise constant function and the collocation method result in the following SLAE
  \[
  t_j \int_{\partial\gamma} u^*(x, \xi) \, d\gamma_x = -\int_{\Omega} u^*(x, \xi_i) f(x) \, d\Omega_x \quad \forall i, j = 1, 2, \ldots, N 
  \]  
  (60)

**Homework**
Using the Somigliana identity compute solution to Laplace problem (56) in the center of a circle of radius $R = 3$, if $f = -4$, $\hat{u} = 9$, $\hat{q} = 6$, $k = 1$.

3. Meshless, FDM - finite difference method

- MLS (moving least squares) approximation - $\varphi(z)$
  - for a given data set of points $S = \{(x_1, y_1), \ldots (x_m, y_m)\}$
  - select a point $z$
  - assume (local) approximation in the form
    \[ L_{XY}(x) = \alpha_1 + \alpha_2(x - z) + \ldots + \alpha_m(x - z)^{m-1} \]
  - assume weights for residuum, eg. $w_i = \frac{1}{(x_i - z)^2 + \varepsilon}$
  - minimize the square of residuum weighted norm $||r||^2 = r^T W r$
    with respect to $a = [\alpha_1, \alpha_2, \ldots, \alpha_m]$
  - by solution of the following SLAE $A^T W A a = A^T W y$
    where $A$ is the Vandermonde matrix, $A = A(z)$, $W = W(z)$, $a = a(z)$
  - $\varphi(z) = L_{XY}(z) = \alpha_1$, note: $\frac{d\varphi}{dz} \neq \frac{dL_{XY}}{dx} |_{x=z}$

- A basis function constructed by MLS approximation for the 3rd out of 6 nodes
Figure 9: A 1D basis function constructed by MLS technique.

Homework

1. Dla zadania (1) i obszaru tam przyjeteego (punkt 3).
   - Napisac wzory na rozwiazanie MES, jego gradient oraz skladowa strumienia \( q_n \) wzdłuż odcinka AD wiedzac, ze \( \alpha_3 = 2, \alpha_4 = -2/3 \).
   - sprawdz dokładnosc spełnienie warunku Neumanna jezeli \( \hat{t}_1 = -1, \hat{t}_2 = 0, \hat{t}_3 = 1 - x \)

2. Dla zadania 1D
   Find \( u(x) \in H_1([a,b]), \text{ such that } u(a) = u(b) = 0 \) and
   \[
   \int_a^b v'AEu' \, dx = \int_a^b vq_0(l - x) \, dx \quad \forall v \in V_0, \ q_0 \in R, \ l = b - a \]  
   (61)
   - Oblicz macierz i wektor dla elementu [2,3] z hierarchiczna funkcja kształtu stopnia 2.
   - Napsz wzory na aproksymacje przemieszczen i siły osiowej
   - Oblicz reakcje stosujac 1 taki element.

3. Stosujac 2 elementy skonczone z liniowymi fukcjami kształtu dla preta
   - oblicz czestosci drgan wlasnych i ich postacie
   - sprawdz ich ortogonalnosc

4. Zadania nieliniowe
   - Wykonaj 1 krok metody Newton-Rahpson dla zadania (48).
   - Znajac \( \varepsilon^p \) w precie oblicz przemieszczenia, odkształcenia (calkowite \( \varepsilon \)) i naprezenia.